UNIT - I MATRIX EIGENVALUE PROBLEM

Defn: -

An arrangement of mn elts. in a rectangular form having an ordered set of 'm' rows & 'n' columns is called a $m \times n$ matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

In short $A = [a_{ij}] = (a_{ij})$, $i = 1, 2, \dots m$; $j = 1, 2, \dots n$.

Here each a_{ij} is called an elt. of the matrix in the i^{th} row & j^{th} column.

Characteristic Equation, Eigen values & Eigen Vectors.

Eigen values & Eigen Vectors: —

Let $A = (a_{ij})$ be a square matrix of order n. If there exists a non-zero column vector

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$ and a scalar $\lambda \ni AX = \lambda X$

Then λ is called the Eigen values of A & X is called Eigen vectors corresponding to λ.

Characteristic Eqn: —

Let A be a square matrix of order n & λ be its Eigen value. Let I be the unit matrix of order n. Then the eqn. $|A - \lambda I| = 0$ is called characteristic eqn. of the matrix A.

Notes:

(i) The determinant $|A-\lambda I|$ is a poly. in λ of degree n and it is called the characteristic polynomial.

(ii) solving the char. eqn. $|A-\lambda I|=0$, we get 'n' values of λ & these 'n' roots are E.Values (or) Latent roots (or) characteristic values of n.

(iii) Corresponding to each value of λ, the eqn. $(A-\lambda I)x=0$ gives a non-zero soln, vector x called E.vector (or) Latent vector (or) Char. vector to the E.Value of λ.

Method to find char. Eqn.:—

Case (i):

If A is a square matrix of order 2 then the char. eqn. of A is $|A-\lambda I|=0$

(or) $\lambda^2 - S_1\lambda + S_2 = 0$

where S_1 = Sum of main diagonal elts

$S_2 = |A|$

Case (ii) :

If A is a square matrix of order 3 then the char. eqn. of A is $|A - \lambda I| = 0$

(or) $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where S_1 = Sum of main diagonal elts.

S_2 = Sum of the minors of main diagonal elts.

$S_3 = |A|$

1) Find the char. eqn. of the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$

Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$

The char. eqn. of A is $\lambda^2 - S_1\lambda + S_2 = 0$.

S_1 = Sum of main diagonal elts

$= 1 + 2 = 3$

$S_2 = |A| = \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = 2.$

$\therefore$ The req. char. eqn. is $\lambda^2 - 3\lambda + 2 = 0$

2) Find the char. eqn. of $\begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$

Let $A = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$

The char eqn. of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Now S_1 = Sum of main diagonal elts

$= 2+1-4 = -1$

S_2 = Sum of minors of the main diagonal elts

$= \begin{vmatrix} 1 & 3 \\ 2 & -4 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ -5 & -4 \end{vmatrix} + \begin{vmatrix} 2 & -3 \\ 3 & 1 \end{vmatrix} = -2$

$S_3 = |A| = \begin{vmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{vmatrix} = 0$

$\therefore$ The reqn. char. eqn is $\lambda^3 + \lambda^2 - 2\lambda = 0$.

Methods to find E.values & E.vectors :

Step 1:- To find the char. eqn. $|A-\lambda I|=0$

Step 2: To solve the char. eqn. we get char. roots called E.v

Step 3: To find E.vectors, solve $(A-\lambda I)x=0$ for diff. values of λ

Note: (to find E.vector)

(i) If all the three rows of matrix $|A-\lambda I|$ are different, then find cofactors of any row of the matrix $|A-\lambda I|$

(ii) If any two rows of matrix $|A-\lambda I|$ is same, then find the cofactors of any one of those two rows.

(iii) If all the three rows are same then we take any one of those three rows.

(iv) If any one of the row is zero then find the cofactor for zero th row.

1) Find the E. values & E. vectors of gn. matrix $\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$

Let $A = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$

Step 1: To find the char. eqn. $|A - \lambda I| = 0$

(or) $\lambda^2 - s_1\lambda + s_2 = 0$

where $s_1 = 0$

$s_2 = -4$

$\therefore$ the char. eqn. is $\lambda^2 - 4 = 0$

Step 2: To find E. values

$\lambda^2 - 4 = 0$

$\lambda = \pm 2$.

$\therefore$ E. values are $-2, 2$.

Step 3: To find E. vector

$(A - \lambda I)x = 0$

$$\left[\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Case (i): If $\lambda = -2$ then

$$\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$3x_1 + x_2 = 0 \quad \text{—} (1)$

$3x_1 + x_2 = 0 \quad \text{—} (2)$

Since (1) & (2) are same, consider any one eqn,

$3x_1 + x_2 = 0$

$3x_1 = -x_2$

$\frac{x_1}{-1} = \frac{x_2}{3}$

$\therefore$ E. vector $X_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

Case (ii): If $\lambda = 2$ then

$$\begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-x_1 + x_2 = 0 \quad \text{—} (1)$

$3x_1 - 3x_2 = 0 \quad \text{—} (2)$

Since (1) & (2) are same, consider any one eqn,

$3x_1 - 3x_2 = 0$

$3x_1 = 3x_2$

$\frac{x_1}{1} = \frac{x_2}{1}$

$\therefore$ E. vector $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$