Boolean Algebra:

A complemented distributive lattice is called a Boolean Algebra

$\vee \to +$
$\wedge \to \cdot$

Boolean Identities

1). Idempotent law: $a \wedge a = a$
$a \vee a = a$

2). Commutative law: $a \wedge b = b \wedge a$
$a \vee b = b \vee a$

3). Identity law: $a \wedge 1 = a$
$a \vee 0 = a$

4). Dominance law: $a \wedge 0 = 0$

5). Absorption law: $a \vee (a \wedge b) = a$
$a \wedge (a \vee b) = a$

6). $a \wedge a' = 0$
$a \vee a' = 0$

7). Demorgan's law
$(a+b)' = a' \cdot b'$
and $(a \cdot b)' = a' + b'$

Theorem : 1

PT in a BA $a = 0$ iff $ab' + a'b = b$

proof:

Assume $a = 0$

Now, $ab' + a'b = 0b' + 1 \cdot b$
$= 0 + b$
$= b$

Conversely,

Assume $ab' + a'b = b$

To prove $a = 0$

Now $ab' = b$ and $a'b = b$
$a' = 1$
$\boxed{a = 0}$

For any BA, prove the following

i). $x \vee y = x \vee z$ and $\bar{x} \vee y = \bar{x} \vee z \Rightarrow y = z$

ii) $x \vee y = 0$ iff $x = 0$ and $y = 0$

iii). $x \leq \bar{y}$ iff $x \wedge y = 0$

iv). $x \wedge y = 1$ iff $x = 1$ and $y = 1$.

Proof:

i). consider $y = y \vee 0$

$= y \vee (x \wedge \bar{x})$

$= (y \vee x) \wedge (y \vee \bar{x})$

$= (x \vee z) \wedge (\bar{x} \vee z)$

$= (x \wedge \bar{x}) \vee z$

$= 0 \vee z$

$y = z$

ii). $x = x \vee 0$

$= x \vee (y \wedge \bar{y})$

$= (x \vee y) \wedge (x \vee \bar{y})$

$= 0 \wedge (x \vee \bar{y})$

$= 0$

and $y = y \vee 0$

$= y \vee (x \wedge \bar{x})$

$= (y \vee x) \wedge (y \vee \bar{x})$

$= 0 \wedge (y \vee \bar{x})$

$= 0$

conversely,

Assume $x = 0$ and $y = 0$.

P. T. $x \vee y = 0$

$x \vee y = 0 \vee 0 = 0$

iii). Assume $x \leq \bar{y} \Rightarrow x \vee \bar{y} = \bar{y}$, $x \wedge \bar{y} = x$

To prove $x \wedge y = 0$

Now $x \wedge y = (x \wedge \bar{y}) \wedge y$

$= x \wedge 0$

$= 0$

conversely, Assume $x \wedge y = 0$

Consider $x = x \wedge 1$

$= x \wedge (y \vee \bar{y})$

$= (x \wedge y) \vee (x \wedge \bar{y})$

$= 0 \vee (x \wedge \bar{y})$

$= x \wedge \bar{y}$

$\Rightarrow x \leq \bar{y}$

iv). Assume that $x \wedge y = 1$

To prove $x = 1$ and $y = 1$.

Now $x = x \wedge 1$

$= x \wedge (y \vee \bar{y})$

$= (x \wedge y) \vee (x \wedge \bar{y})$

$= 1 \vee (x \wedge \bar{y})$

$= 1$

and $y = y \wedge 1$

$= y \wedge (x \vee \bar{x})$

$= (y \wedge x) \vee (y \wedge \bar{x})$

$= 1 \vee (y \wedge \bar{x})$

$= 1$

conversely, Assume $x = 1$ and $y = 1$.

To prove $x \wedge y = 1$

$x \wedge y = 1 \wedge 1 = 1$.