Normal subgroup:

Let $H$ be a subgroup of $G$ under $*$. Then $H$ is said to be a normal subgroup of $G$, for evy. $x \in G$ and for $h \in H$ if

$$x * h * x^{-1} \in H$$

and $x * H * x^{-1} \subseteq H$

Alternatively, a subgroup $H$ of $G$ is called a normal subgroup of $G$ if $x * h = h * x$, $\forall x \in G$.

Theorem: 1

The intersection of any 2 normal subgroups is a normal subgroup.

Proof:

Let $H$ and $K$ be the 2 normal subgroups.

$\Rightarrow H$ and $K$ are subgroups of $G$.

$\Rightarrow H \cap K$ is a subgroup of $G$ (Already proved)

Now we've to prove that $H \cap K$ is normal.

Let $x \in G$ and $h \in H \cap K$

$x \in G$ and $h \in H$ and $h \in K$

$\Rightarrow x \in G$, ~~and~~ $h \in H$ and $x \in G$, $h \in K$

$\Rightarrow$ $\underbrace{x * h * x^{-1} \in H}_{(1)}$ and $\underbrace{x * h * x^{-1} \in K}_{(2)}$ ($\because H$ and $K$ are normal subgroups)

From (1) and (2), we get

$$x * h * x^{-1} \in H \cap K$$

$\Rightarrow H \cap K$ is a normal subgroup of $G$.

Theorem 2:

Let $G$ and $G'$ be any two groups with identity elt. $e$ and $e'$ resply. If $f: G \to G'$ be a homomorphism, then $\ker(f)$ is a normal subgroup.

Proof:

Let $K = \ker f = \{x \in G / f(x) = e'\}$

we know that $\ker(f)$ is a subgroup of $G$

Now we've to prove that $\ker(f)$ is normal.

To prove $x * h * x^{-1} \in K$

Let $x \in G$ and $h \in K$

$\therefore f(x * h * x^{-1}) = f(x) * f(h) * f(x^{-1})$ $\because f$ is a homo.

$= f(x) * e' * f(x^{-1})$ $[\because h \in K = \ker f]$

$= f(x) * f(x^{-1})$

$= f(x * x^{-1})$

$= f(e)$

$= e'$

$f(x * h * x^{-1}) = e'$

$\Rightarrow x * h * x^{-1} \in K$

$\therefore K = \ker f$ is a normal subgroup of $G$.

Theorem: 3 Fundamental Theorem of Homomorphism

Every homomorphic image of a group $G$ is isomorphic to some quotient group of $G$.

(or)

Let $f: G \to G'$ be a onto homomorphism of groups with kernel $K$. Then $G/K \simeq G'$.

Proof:

Let $f: G \to G'$ be a homomorphism

Let $K$ be the kernel of this homo.

Clearly $K$ is a normal subgroup of $G$.

To prove $G/K$ is isomorphic $G/K \simeq G'$.

i). To prove $\phi$ is well defined.

Let $\phi: G/K \to G'$ by $\phi(K*a) = f(a)$.

Consider, $K*a = K*b$

$\Rightarrow a*b^{-1} \in K$

$\Rightarrow f(a*b^{-1}) = e'$

$f(a) * f(b^{-1}) = e'$

$f(a) * [f(b)]^{-1} = e'$

$f(a) * [f(b)]^{-1} * f(b) = e' * f(b)$

$f(a) * e = e' * f(b)$

$f(a) = f(b)$

$\phi(K*a) = \phi(K*b)$

$\therefore \phi$ is well defined.

ii). To prove $\phi$ is 1-1.

ie., $\phi(K*a) = \phi(K*b) \Rightarrow K*a = K*b$

consider $\phi(K*a) = \phi(K*b)$

$$f(a) = f(b)$$
$$f(a) * f(b^{-1}) = f(b) * f(b^{-1})$$
$$f(a * b^{-1}) = f(b * b^{-1})$$
$$= f(e)$$
$$= e'$$
$$\Rightarrow a * b^{-1} \in K$$
$$K * a = K * b$$
$\therefore \phi$ is 1-1.

iii). To prove $\phi$ is onto.

Let $b \in G'$

Since $f$ is onto. $\exists$ an elt. $a \in G$ such that $f(a) = b$.

$$\Rightarrow f(a) = \phi(K * a) = b$$
$\therefore \phi$ is onto.

iv). To prove $\phi$ is a homo.

Now,
$$\phi(K * a * K * b) = \phi(K * a * b)$$
$$= f(a * b)$$
$$= f(a) * f(b)$$
$$= \phi(K * a) * \phi(K * b)$$
$\therefore \phi$ is a homo.

Since $\phi$ is 1-1 & onto, homo.

$\therefore \phi$ is an isomorphism b/w $G/K$ and $G'$

$$\Rightarrow G/K \simeq G'.$$