SNS COLLEGE OF TECHNOLOGY
Coimbatore

TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS

UNIT – II

FOURIER TRANSFORMS

PART A

1. The infinite transform of a function f(x) is defined by

   a. $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x).e^{isx} dx$ b. $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(x).e^{isx} dx$

   c. $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x).e^{-isx} dx$ d. $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(x).e^{-isx} dx$

   Ans : a

2. The infinite Fourier sine transform of f(x) is

   a. $F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} f(x).\sin sx.ds$ b. $F_s[f(x)] = \sqrt{\frac{1}{\pi}} \int_{-\infty}^{\infty} f(x).\sin sx.ds$

   c. $F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x).\sin sx.dx$ d. $F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F(s).\sin sx.dx$

   Ans : c

3. The infinite Fourier cosine transform of f(x) is

   a. $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} f(x).\cos sx.dx$ b. $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x).\cos sx.dx$

   c. $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F(s).\cos sx.dx$ d. $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} F(s).\cos sx.dx$

   Ans : b

4. The inverse Fourier cosine transform of f(x) is

   a. $f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_c[f(x)].\cos sx.ds$ b. $f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_c[f(x)].\cos sx.dx$

c. $f(x)=\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} F_c[f(x)].\cos sx.ds$ d. $f(x)=\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} F_c[f(x)].\cos sx.dx$

Ans : a

5. If $F_s[f(x)]=F_s(s)$ and $F_s[g(x)]=G_s(s)$,then

a. $F_s[af(x)+bg(x)]=aF_s(s)+bG_s(s)$ b. $F_s[bf(x)+ag(x)]=aF_s(s)+bG_s(s)$

c. $F_s[af(x)+bg(x)]=aF_s(s)-bG_s(s)$ d. None of the above

Ans : a

6. If $F[f(x)]=F(s)$, then

a. $F[f(ax)]=aF\left(\frac{s}{a}\right), a>0$ b. $F[f(ax)]=\frac{1}{a}F\left(\frac{s}{a}\right), a>0$

c. $F[f(ax)]=\frac{1}{s}F\left(\frac{s}{a}\right), a>0$ d. $F[f(ax)]=\frac{1}{a}F\left(s\right), a>0$

Ans : b

7. If $F_s[f(x)]=F_s(s)$, then

a. $F_c[f(ax)]=\frac{1}{a}F_c\left(\frac{s}{a}\right), a>0$ b. $F_c[f(ax)]=-\frac{1}{a}F_c\left(\frac{s}{a}\right), a>0$

c. $F_c[f(ax)]=\frac{1}{s}F_c\left(\frac{s}{a}\right), a>0$ d. $F_c[f(ax)]=F_c\left(\frac{s}{a}\right), a>0$

Ans : a

8. If $F_s[f(x)]=F_s(s)$, then

a. $F_s[f(x).\cos ax]=\frac{1}{2}[F_s(s+a)+F_s(s-a)]$

b. $F_s[f(x).\cos ax]=\frac{1}{2}[F_s(s+a)-F_s(s-a)]$

c. $F_s[f(x).\cos ax]=\frac{1}{2}[F_c(s+a)+F_c(s-a)]$

d. $F_s[f(x).\cos as]=\frac{1}{2}[F_s(s+a)+F_s(s-a)]$

Ans : a

9. If $F_c[f(x)]=F_c(s)$, then

a. $F_c[f(x).\cos ax]=\frac{1}{2}[F_c(s+a)+F_c(s-a)]$

b. $F_c[f(x).\cos ax] = \frac{1}{2}[F_c(s+a) - F_c(s-a)]$

c. $F_c[f(x).\sin ax] = \frac{1}{2}[F_c(s+a) + F_c(s-a)]$

d. None of the above

Ans : a

10. If $F[f(x)] = F(s)$, then

a. $F_s[f'(x)] = -sF_c(s)$

b. $F_c[f'(x)] = -sF_s(s)$

c. $F_s[f(x)] = -sF_c(s)$

d. $F_s[f(x)] = sF_c'(s)$

Ans : a

11. Parseval's identity for Fourier sine transform is

a. $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F_s(s)|^2 ds$

b. $\int_{0}^{\infty} f(x) dx = \int_{0}^{\infty} F_s(s) ds$

c. $\int_{0}^{\infty} |f(x)|^2 dx = \int_{0}^{\infty} |F_s(s)|^2 ds$

d. $\int_{0}^{\infty} |f(x)|^2 dx = \int_{0}^{\infty} F_s(s) ds$

Ans : c

12. The convolution theorem for Fourier transform is

a. $F\left[(f+g)x\right] = F(s).G(s)$

b. $F\left[(f*g)x\right] = F(s).G(s)$

c. $F\left[(f \pm g)x\right] = F(s) \pm G(s)$

d. $F\left[(f*g)x\right] = F(s) + G(s)$

Ans : b

13. The Fourier sine transform of e^{-x}

a. $\sqrt{\frac{2}{\pi}} \frac{s}{s^2+1}$

b. $\sqrt{\frac{1}{\pi}} \frac{s}{s^2+1}$

c. $\sqrt{\frac{2}{\pi}} \frac{s}{s^2-1}$

d. $\sqrt{\frac{2}{\pi}} \frac{s^2}{s^2-1}$

Ans : a

14. The Fourier sine transform of e^{-ax}

a. $\sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2}$

b. $\sqrt{\frac{2}{\pi}} \frac{a}{s^2+a}$

c. $\sqrt{\frac{2}{\pi}} \frac{a}{s^2-a^2}$

d. $\sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}$

Ans : d

15. The convolution of two function f(x) and g(x) is

a. $f + g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)g(x-t)dt$

b. $f * g = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(t)g(x-t)dt$

c. $f * g = \int_{-\infty}^{\infty} f(t)g(x-t)dt$

d. $f * g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)g(x-t)dt$

Ans : d

PART B
TWO MARKS

1. Prove that $F[f(ax)] = \frac{1}{a} F\left(\frac{s}{a}\right), a > 0$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax)e^{isx}dx$$

put $ax = y$

$adx = dy$

$dx = \frac{dy}{a}$

when $x = -\infty, y = -\infty$ and $x = -\infty, y = \infty$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)e^{is\left(\frac{y}{a}\right)} \frac{dy}{a}$$

$$= \frac{1}{a}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)e^{is\left(\frac{y}{a}\right)}dy$$

$$= \frac{1}{a}F\left(\frac{s}{a}\right)$$

2. If F(s)=F[f(x)],then prove that $F[xf(x)] = (-i)\frac{d[F(s)]}{ds}$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$\frac{d}{ds}[F(s)] = \frac{d}{ds}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx}dx\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\frac{\partial}{\partial s}\left(e^{isx}\right)dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)(ix)e^{isx}dx$$

$$= iF[xf(x)]$$

$$\therefore F[xf(x)] = (-i)\frac{d}{ds}[F(s)]$$

3. If $F_S(s)$ is the Fourier sine transform of f(x), show that

$$F_S[f(x)\cos ax] = \frac{1}{2}[F_S(s+a) + F_S(s-a)]$$

$$F_S[f(x)\cos ax] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)\cos ax \sin sx dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \int_0^{\infty} f(x)\{\sin(a+s)x - \sin(a-s)x\} dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{1}{2}\int_0^{\infty} f(x)\sin(a+s)xdx - \frac{1}{2}\int_0^{\infty} f(x)\sin(a-s)xdx\right]$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\{\sin(a+s)x\}\,dx - \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\{\sin(a-s)x\}\,dx\right]$$

$$= \frac{1}{2}\left[F_S(s+a) + F_S(s-a)\right]$$

4. Define self reciprocal function and give example.

If the transform of f(x) is equal to f(s),then the function f(x) is called self-reciprocal.Example

$f(x) = e^{-\frac{x^2}{2}}$ **is self reciprocal under Fourier cosine transform.**

5. State Parseval's identity on complex Fourier Transforms.

$$\int_{0}^{\infty} |f(x)|^2 dx = \int_{0}^{\infty} |F(s)|^2 \, ds$$

$$\frac{d^2}{ds^2}[F(s)] = \frac{1}{\sqrt{2\pi}} \frac{d}{ds} \int_{-\infty}^{\infty} f(x).ix\left(e^{isx}\right)dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial}{\partial s}\left[f(x).ix\left(e^{isx}\right)\right]dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)(ix)^2\left(e^{isx}\right)dx$$

$$= (i)^2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 f(x)\left(e^{isx}\right)dx$$

6. **Find the Fourier Transform of** $f(x) = \begin{cases} 1 & \text{in} \quad |x| < a \\ 0 & \text{in} \quad |x| > a \end{cases}$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} f(x)e^{isx}dx + \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} f(x)e^{isx}dx + \frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} f(x)e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} f(x)e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} 1.e^{isx}dx = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isx}}{is} \right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{is} \left(e^{isa} - e^{-isa} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{is} 2i \sin sa = \sqrt{\frac{2}{\pi}} \frac{\sin sa}{s}$$

7. Find the Fourier sine transform of e^{-x}

The Fourier sine transform of f(x) is given by

$$F_S[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx dx$$

$$\text{Here} \quad e^{-x} = e^{-|x|} \quad \text{for} \quad x > 0$$

$$\therefore F_S\left[e^{-x}\right] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x} \sin sx dx$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2 + 1} \right)$$

8. Find the Fourier sine transform of e^{ax} **.**

$$F_S\left[e^{ax}\right]=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax}\sin sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right)$$

9. State the convolution theorem for Fourier transforms

If F(s) and G(s) are the Fourier transform of f(x) and g(x) respectively then the fourier transform of the convolution of f(x) and g(x) is the product of their Fourier Transform.

$$\text{ie}\quad F\left[(f*g)(x)\right]=F(s).G(s)$$

$$\text{ie}\quad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(f*g)(x)e^{isx}dx=\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx\right\}.\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)e^{isx}dx\right\}$$

10. Write the Fourier Transform pair.

If f(x) is a given function, then $F\left[f(x)\right]$ **and** $F^{-1}\left[F(f(x))\right]$ **are called Fourier transform pair, where**

$$F\left[f(x)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-isx}dx$$

$$F^{-1}\left[F(f(x))\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(f(x))e^{isx}ds$$

PART C
MODEL QUESTIONS

1. Find the Fourier Transform of $f(x)=\begin{cases}1, & |x|<a\\ 0, & |x|>a\end{cases}$ and hence find $\int_0^\infty \frac{\sin x}{x}dx$

2. Find the Fourier Transform of $e^{\frac{-x^2}{2}}$

3. Verify the convolution theorem under Fourier Transform for $f(x)=g(x)=e^{-x^2}$

4. Evaluate $\int_0^\infty \frac{dx}{\left(x^2+a^2\right)^2}$ using Parseval's identity.

5. Find the Fourier Transform of $f(x)=\begin{cases}1-|x|, & |x|<1\\ 0, & |x|>1\end{cases}$ and hence find $\int_0^\infty \frac{\sin^4 t}{t^4}dt$

6. Find the Fourier Transform of $f(x)=\begin{cases}a^2-x^2, & |x|<a\\ 0, & |x|>a>0\end{cases}$ and hence deduce that

$$\int_0^\infty \frac{\sin t - t\cos t}{t^3}dt=\frac{\pi}{4}$$

7. Find the Fourier cosine and sine transforms of $f(x)=e^{-ax}, a>0$ and hence deduce the inverse formula.

8. Find the Fourier cosine transform of $e^{-a^2x^2}$,a>0. Hence show that the function $e^{\frac{-x^2}{2}}$

9. Find the Fourier cosine transform of $f(x)=\begin{cases}x, & 0<x<1\\ 2-x, & 1<x<2\\ 0, & x>2\end{cases}$

10. Derive the Parseval's identity for Fourier Transforms.

11. State and prove convolution theorem on Fourier Transform.

12. Find the Fourier sine and cosine transform of x^{n-1} and hence prove $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier sine and cosine transforms.