Properties of Fourier Transform, FST and FCT:

I. Linear Property:

→ **FT** $F[af(x)+bg(x)] = a\,F[f(x)] + b\,F[g(x)]$ where a and b are real numbers.

proof:

$$F[af(x)+bg(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[af(x)+bg(x)]\,e^{isx}\,dx$$

$$= \frac{a}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx + b\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)\,e^{isx}\,dx$$

$$= a\,F[f(x)] + b\,F[g(x)]$$

→ **FST** $F_s[af(x)+bg(x)] = a\,F_s[f(x)] + b\,F_s[g(x)]$

proof:

$$F_s[af(x)+bg(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty}[af(x)+bg(x)]\sin sx\,dx$$

$$= a\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx\,dx + b\sqrt{\frac{2}{\pi}}\int_0^{\infty} g(x)\sin sx\,dx$$

$$= a\,F_s[f(x)] + b\,F_s[g(x)]$$

IIIly FCT $F_c[af(x)+bg(x)] = a\,F_c[f(x)] + b\,F_c[g(x)]$

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2. change of scale property:

For any non-zero real a, $F[f(ax)] = \frac{1}{|a|}F\left[\frac{s}{a}\right]$, $a>0$

proof:

WKT, $F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx$

Now, $F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax)\, e^{isx}\, dx$

put $t = ax$ | when $x = -\infty \Rightarrow t = -\infty$

$\frac{dt}{dx} = a$ | $x = \infty \Rightarrow t = \infty$

$\Rightarrow dx = \frac{dt}{a}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{is\frac{t}{a}}\, \frac{dt}{a}$$

$$= \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i\left(\frac{s}{a}\right)t} \, dt$$

$$= \frac{1}{a} F\left[\frac{s}{a}\right]$$

3]. Shifting Property:

i). $F[f(x-a)] = e^{ias} F(s)$

ii). $F[e^{iax} f(x)] = F(s+a)$

proof:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{isx}\, dx$$

i). Now, $F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a)\, e^{isx}\, dx$

put $t = x - a$ | $x = -\infty \Rightarrow t = -\infty$

$dt = dx$ | $x = \infty \Rightarrow t = \infty$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{is(t+a)}\, dt$$

$$= \frac{e^{isa}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt = e^{ias} F(s)$$

ii). $F[e^{iax} f(x)] = F[s+a]$

Proof:

Now $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$$F[e^{iax} f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(s+a)x} dx$$

$$= F(s+a)$$

4). Modulation Property:

$\underrightarrow{\text{PT}}$ If $F(s)$ is the Fourier transform of $f(x)$,

then $F[f(x) \cos ax] = \frac{1}{2}[F(s+a) + F(s-a)]$

proof:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

Now, $F[f(x) \cos ax] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos ax \, e^{isx} dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left[\frac{e^{iax} + e^{-iax}}{2} \right] e^{isx} dx$$

$$= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left[e^{i(s+a)x} + e^{i(s-a)x} \right] dx$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{i(s+a)x}\, dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{i(s-a)x}\, dx\right]$$

$$= \frac{1}{2}\left[F(s+a) + F(s-a)\right]$$

FST →

$$\cancel{F_S[f(x) \sin ax] = \frac{1}{2}}$$

$$F_S[f(x) \cos ax] = \frac{1}{2}\left[F_S(s+a) + F_S(s-a)\right]$$

Proof:

WKT $F_S[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x) \sin sx\, dx$

Now, $F_S[f(x)\cos ax] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x) \cos ax \sin sx\, dx$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x) \sin \underbrace{sx}_{A} \cos \underbrace{ax}_{B}\, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x) \frac{1}{2}\left[\sin \overset{(sx+ax)}{(A+B)} + \sin(sx-ax)\right] dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x) \sin(s+a)x\, dx + \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x) \sin(s-a)x\, dx\right]$$

$$= \frac{1}{2}\left[F_S(s+a) + F_S(s-a)\right]$$

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5]. $F[x^n f(x)] = (-i)^n \frac{d^n}{ds^n} F(s)$

Proof :

$$F[s] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

Differentiating both sides 'n' times with respect to 's'

$$\frac{d^n}{ds^n} F(s) = \frac{1}{\sqrt{2\pi}} \frac{d^n}{ds^n} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^n}{\partial s^n} [f(x) e^{isx}]\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, (ix)^n\, e^{isx}\, dx$$

$$= i^n \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, x^n\, e^{isx}\, dx$$

$$= i^n F[x^n f(x)]$$

$$\Rightarrow F[x^n f(x)] = \frac{1}{(i)^n} \frac{d^n}{ds^n} F(s)$$

$$= (-i)^n \frac{d^n}{ds^n} F(s)$$

6]. i). $F[f'(x)] = -is F(s)$ if $f(x) \to 0$ as $x \to \pm\infty$

ii). $F[f^{(n)}(x)] = (-i)^n s^n F(s)$ if $f(x), f'(x), \dots$

$f^{(n-1)}(x) \to 0$ as $x \to \pm\infty$

7]. $F[\overline{f(x)}] = \overline{F(-s)}$

8]. i) $F_s[x f(x)] = -\frac{d}{ds} F_c[f(x)]$

ii) $F_c[x f(x)] = \frac{d}{ds} F_s[f(x)]$

Problems on properties.

1]. Find the F.S. and F.C.T. of $x e^{-ax}$.

Soln.:

By property,

$$F_s[x f(x)] = -\frac{d}{ds} F_c[f(x)]$$

$$F_s[x e^{-ax}] = -\frac{d}{ds} F_c[e^{-ax}]$$

$$= -\frac{d}{ds} \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}$$

$$F_s[x e^{-ax}] = \frac{\sqrt{2}}{\pi} + \frac{a2s}{(a^2+s^2)^2}$$

and

$$F_c[x f(x)] = \frac{d}{ds} F_s[f(x)]$$

$$F_c[x e^{-ax}] = \frac{d}{ds} F_s[e^{-ax}]$$

$$= \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{(s^2+a^2)(1) - s(2s)}{(s^2+a^2)^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

H.W. Find the FCT of $e^{-a^2x^2}$ and hence find $F_c[x e^{-a^2x^2}]$.