Find the Fourier series for

$$f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi < x < 0 \\ 1 - \frac{2x}{\pi}, & 0 < x < \pi \end{cases}$$

Soln:- consider

$$\phi_1(x) = 1 + \frac{2x}{\pi}$$

$$\phi_2(x) = 1 - \frac{2x}{\pi}$$

$$\phi_1(-x) = 1 + \frac{2(-x)}{\pi}$$

$$= 1 - \frac{2x}{\pi}$$

$$= \phi_2(x)$$

$$\phi_2(-x) = 1 - \frac{2(-x)}{\pi}$$

$$= 1 + \frac{2x}{\pi}$$

$$= \phi_1(x)$$

$\therefore$ $f(x)$ is even Function.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)dx.$$

$$= \frac{2}{\pi}\left[x - \frac{2}{\pi}\frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi - \frac{\pi^{\cancel{2}}}{\cancel{\pi}}\right]$$

$$= \frac{2}{\pi}(\pi - \pi)$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{\text{Even}} \underbrace{\cos nx}_{\text{Even}}\, dx.$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)\cos nx\, dx.$$

$u = 1 - \frac{2x}{\pi}$ $\qquad\qquad$ $v = \cos nx.$

$u' = -\frac{2}{\pi}$ $\qquad\qquad$ $v_1 = \frac{\sin nx}{n}$

$u'' = 0$ $\qquad\qquad$ $v_2 = -\frac{\cos nx}{n^2}$

$$= \frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\frac{\sin nx}{n} - \left(\frac{-2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[0 - \frac{2}{\pi}\frac{\cos n\pi}{n^2} - 0 + \frac{2}{\pi}\frac{\cos 0}{n^2}\right]$$

$$= \frac{2}{\pi}\left[-\frac{2(-1)^n}{\pi n^2} + \frac{2}{\pi n^2}\right]$$

$$= \frac{2}{\pi}\left(\frac{2}{\pi n^2}\right)\left[-(-1)^n + 1\right]$$

$$a_n = \frac{4}{\pi^2 n^2}\left[1-(-1)^n\right]$$

$\therefore$ The Fourier series is

$$f(x) = 0 + \sum_{n=1}^{\infty}\frac{4}{\pi^2 n^2}\left[1-(-1)^n\right]\cos nx.$$

$$= \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1-(-1)^n\right]\cos nx.$$

Find the Fourier series for

$$f(x) = \begin{cases} L+x, & -L < x < 0 \\ L-x, & 0 < x < L \end{cases}$$

Soln:-

$$\phi_1(x) = L+x$$
$$\phi_1(-x) = L-x$$
$$= \phi_2(x).$$

$$\phi_2(x) = L-x$$
$$\phi_2(-x) = L+x$$
$$= \phi_1(x)$$

$\therefore$ $f(x)$ is even function.

$\therefore$ The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx.$$

$$= \frac{2}{L}\int_0^L f(x)\,dx$$

$$= \frac{2}{L}\int_0^L (L-x)\,dx.$$

$$= \frac{2}{L}\left[Lx - \frac{x^2}{2}\right]_0^L$$

$$= \frac{2}{L}\left[L^2 - \frac{L^2}{2}\right]$$

$$= \frac{2}{\cancel{L}} L^{\cancel{2}}\left(1 - \frac{1}{2}\right)$$

$$= \cancel{2}L\left(\frac{1}{\cancel{2}}\right)$$

$$\boxed{a_0 = L}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx.$$

$$= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\int_0^L (L-x)\cos\left(\frac{n\pi x}{L}\right)dx.$$

$u = L - x$

$u' = -1$

$u'' = 0$

$v = \cos\left(\frac{n\pi x}{L}\right)$

$v_1 = \dfrac{+\sin\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}}$

$v_2 = -\dfrac{\cos\left(\frac{n\pi x}{L}\right)}{\frac{n^2\pi^2}{L^2}}$

$$= \frac{2}{L}\left[(L-x)\frac{\sin\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}} - (-1)\left(-\frac{\cos\left(\frac{n\pi x}{L}\right)}{\frac{n^2\pi^2}{L^2}}\right)\right]_0^L$$

$$= \frac{2}{L}\left[0 - \frac{\cos\left(\frac{n\pi \cancel{L}}{\cancel{L}}\right)}{\frac{n^2\pi^2}{L^2}} - 0 + \frac{\cos 0}{\frac{n^2\pi^2}{L^2}}\right]$$

$$= \frac{2}{\cancel{L}}\frac{L^{\cancel{2}}}{n^2\pi^2}\left[-\cos n\pi + 1\right]$$

$$a_n = \frac{2L}{n^2\pi^2}\left[-(-1)^n + 1\right]$$

$\therefore$

$$f(x) = \frac{L}{2} + \sum_{n=1}^{\infty}\frac{2L}{n^2\pi^2}\left[1-(-1)^n\right]\cos\left(\frac{n\pi x}{L}\right)$$

$$= \frac{L}{2} + \frac{2L}{\pi^2}\sum_{n=1}^{\infty}\left(1-(-1)^n\right)\cos\left(\frac{n\pi x}{L}\right)$$