(2) Find the Fourier series for the function $f(x) = \frac{(\pi - x)^2}{2}$ in $0 \le x \le 2\pi$.

Soln:- $f(x) = \frac{(\pi - x)^2}{2}$

Fourier series for the function $f(x)$ in the interval $[0, 2\pi]$ is.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

To find a_0:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi - x)^2}{2}\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\pi - x)^2\,dx$$

$$= \frac{1}{2\pi}\left[\frac{(\pi - x)^3}{-3}\right]_0^{2\pi}$$

$$= \frac{-1}{6\pi}\left[(\pi - 2\pi)^3 - \pi^3\right]$$

$$= -\frac{1}{6\pi}\left[(-\pi^3) - \pi^3\right]$$

$$= \frac{-1}{\cancel{6\pi}_{3}}\left(-\cancel{2}\pi^3\right) = \frac{\pi^2}{3}$$

To find a_n:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi-x)^2}{2}\cos nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\pi-x)^2\cos nx\,dx$$

$u = (\pi-x)^2$ $\qquad$ $v = \cos nx$

$u' = 2(\pi-x)(-1)$ $\qquad$ $v_1 = +\frac{\sin nx}{n}$

$= -2(\pi-x)$

$u'' = -2(-1)$ $\qquad$ $v_2 = -\frac{\cos nx}{n^2}$

$= 2$ $\qquad$ $v_3 = -\frac{\sin nx}{n^3}$

$u''' = 0$

$$a_n = \frac{1}{2\pi}\left[(\pi-x)^2\frac{\sin nx}{n} - \left[-2(\pi-x)\right]\left[-\frac{\cos nx}{n^2}\right] + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[0 - 2(-\pi)\frac{\cos n2\pi}{n^2} - 0 - 0 + 2(\pi)\frac{\cos 0}{n^2} + 0\right]$$

$$= \frac{1}{2\pi}\left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right]$$

$$= \frac{1}{\cancel{2\pi}}\left[\frac{\overset{2}{\cancel{4\pi}}}{n^2}\right]$$

$$\boxed{a_n = \frac{2}{n^2}}$$

To find bn:

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi-x)^2}{2}\sin nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\pi-x)^2 \sin nx\,dx$$

$u = (\pi-x)^2$ $\qquad$ $v = \sin nx$

$u' = 2(\pi-x)(-1)$ $\qquad$ $v_1 = -\frac{\cos nx}{n}$

$= -2(\pi-x)$

$u'' = -2(-1)$ $\qquad$ $v_2 = -\frac{\sin nx}{n^2}$

$= 2$ $\qquad$ $v_3 = +\frac{\cos nx}{n^3}$

$u''' = 0$

$$= \frac{1}{2\pi}\left[-(\pi-x)^2\frac{\cos nx}{n} - [-2(\pi-x)]\left[-\frac{\sin nx}{n^2}\right] + 2\frac{\cos nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[-(\pi-2\pi)^2\frac{\cos n2\pi}{n} - 0 + \frac{2\cos n2\pi}{n^3} + \pi^2\frac{\cos 0}{n} + 0 - \frac{2\cos 0}{n^3}\right]$$

$$= \frac{1}{2\pi}\left[-\frac{(-\pi)^2}{n} + \cancel{\frac{2}{n^3}} + \frac{\pi^2}{n} - \cancel{\frac{2}{n^3}}\right]$$

$$= \frac{1}{2\pi}\left(\cancel{-\frac{\pi^2}{n}} + \cancel{\frac{\pi^2}{n}}\right)$$

$$\boxed{b_n = 0}$$

The fourier Series is

$$f(x) = \frac{\frac{\pi^2}{3}}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2} \cos nx + 0$$

$$= \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2}{n^2} \cos nx$$

Interval: $[0, 2l]$

(3) Find the Fourier Series for the function $f(x) = x^2$ in $(0, 2l)$.

Solution: $f(x) = x^2$ in $(0, 2l)$.

The Fourier Series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

To Find a_0:

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$= \frac{1}{l}\int_0^{2l} x^2\,dx$$

$$= \frac{1}{l}\left[\frac{x^3}{3}\right]_0^{2l}$$

$$= \frac{1}{l}\left[\frac{8l^3}{3} - 0\right]$$

$$= \frac{1}{\cancel{l}}\left[\frac{8l^{\cancel{3}}}{3}\right]$$

$$\boxed{a_0 = \frac{8l^2}{3}}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\int_0^{2l} x^2 \cos\left(\frac{n\pi x}{l}\right)dx.$$

$u = x^2$ $\qquad$ $v = \cos\left(\frac{n\pi x}{l}\right)$

$u' = 2x$ $\qquad$ $v_1 = \frac{\sin\left(\frac{n\pi x}{l}\right)}{(n\pi/l)}$

$u'' = 2$ $\qquad$ $v_2 = -\frac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}$

$u''' = 0$ $\qquad$ $v_3 = -\frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}$

$$a_n = \frac{1}{l}\left[x^2 \frac{\sin\left(\frac{n\pi x}{l}\right)}{(n\pi/l)} - 2x\left[-\frac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right] + 2\left[-\frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}\right]\right]_0^{2l}$$

$$= \frac{1}{l}\left[0 + 2(2l)\frac{\cos\left(\frac{n\pi 2l}{l}\right)}{\left(\frac{n\pi}{l}\right)^2} - 0 - 0 - 0 + 0\right]$$

$$= \frac{1}{l}\left[4l \cos(n2\pi) \frac{l^2}{n^2\pi^2}\right]$$

$$= \frac{1}{\cancel{l}}\left[\frac{4l^{\cancel{3}}}{n^2\pi^2}\right]$$

$$\boxed{a_n = \frac{4l^2}{n^2\pi^2}}$$

<u>To find bn:</u>

$$b_n = \frac{1}{l}\int_0^{2l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{1}{l}\int_0^{2l} x^2 \sin\left(\frac{n\pi x}{l}\right) dx.$$

$$u = x^2 \qquad v = \sin\left(\frac{n\pi x}{l}\right)$$
$$u' = 2x \qquad v_1 = -\frac{\cos\left(\frac{n\pi x}{l}\right)}{(n\pi/l)}$$
$$u'' = 2 \qquad v_2 = -\frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}$$
$$u''' = 0 \qquad v_3 = \frac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}$$

$$b_n = \frac{1}{l}\left[-x^2\frac{\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}} - 2x\left[-\frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right] + 2\frac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}\right]_0^{2l}$$

$$= \frac{1}{l}\left[-4l^2\frac{\cos\left(\frac{n\pi 2l}{l}\right)}{\frac{n\pi}{l}} + 2(2l)\left[\frac{\sin\left(\frac{n\pi 2l}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right] + 2\frac{\cos\left(\frac{n\pi 2l}{l}\right)}{\left(\frac{n\pi}{l}\right)^3} + 0 - 0 - 2\frac{\cos 0}{\left(\frac{n\pi}{l}\right)^3}\right]$$

$$= \frac{1}{l}\left[\frac{-4l^2}{(n\pi/l)} + 0 + \frac{2}{\left(\frac{n\pi}{l}\right)^3} - \frac{2}{\left(\frac{n\pi}{l}\right)^3}\right]$$

$$= \frac{1}{\cancel{l}}\left[-4l^2 \times \frac{\cancel{l}}{n\pi}\right]$$

$$\boxed{b_n = \frac{-4l^2}{n\pi}}$$

$\therefore$ The Fourier Series is

$$f(x) = \frac{\frac{8l^2}{3}}{2} + \sum_{n=1}^{\infty}\left(\frac{4l^2}{n^2\pi^2}\right)\cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty}\left(\frac{-4l^2}{n\pi}\right)\sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{4l^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4l^2}{n^2\pi^2}\right)\cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty}\left(\frac{-4l^2}{n\pi}\right)\sin\left(\frac{n\pi x}{l}\right).$$