Half Range series :-

Cosine series :

The half-range cosine series of $f(x)$ defined in the interval $0<x<\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx .$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$ and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$.

Find the Half range series for $f(x) = \begin{cases} \frac{\pi x}{4}, & 0<x<\pi/2 \\ \frac{\pi}{4}(\pi-x), & \pi/2<x<\pi \end{cases}$

(i) Half-range cosine series:

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$.

Now $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \frac{\pi x}{4}\,dx + \int_{\pi/2}^{\pi} \frac{\pi}{4}(\pi-x)\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{4}\left[\frac{x^2}{2}\right]_0^{\pi/2} + \frac{\pi}{4}\left[\pi x - \frac{x^2}{2}\right]_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{4}\cdot\frac{\pi^2}{8} + \frac{\pi}{4}\left(\left(\pi^2 - \frac{\pi^2}{2}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^2}{8}\right)\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{32} + \frac{\pi^3}{8} - \frac{3\pi^3}{32}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{8} - \frac{\cancel{2}\pi^3}{\cancel{32}_{16}}\right]$$

$$= \frac{\cancel{2}}{\cancel{\pi}}\cdot\frac{\pi^{\cancel{3}\,2}}{\cancel{16}\,8} = \frac{\pi^2}{8}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \frac{\pi}{4} x \cos nx \, dx + \int_{\pi/2}^{\pi} \frac{\pi}{4}(\pi - x)\cos nx \, dx\right]$$

$$= \frac{2}{\pi}\left\{\frac{\pi}{4}\left[x\frac{\sin nx}{n} - 1\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi/2} + \frac{\pi}{4}\left[(\pi - x)\frac{\sin nx}{n} - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{\cancel{2}}{\cancel{\pi}} \times \frac{\cancel{\pi}}{\cancel{4}\,2}\left\{\left[\frac{\pi}{2}\frac{\sin n\pi/2}{n} + \frac{\cos n\pi/2}{n^2} - \frac{\cos 0}{n^2}\right] + \left[0 - \frac{\cos n\pi}{n^2} - \left(\frac{\pi}{2}\frac{\sin n\pi/2}{n} - \frac{\cos n\pi/2}{n^2}\right)\right]\right\}$$

$$= \frac{1}{2}\left\{\cancel{\frac{\pi}{2}\frac{\sin n\pi/2}{n}} + \frac{\cos n\pi/2}{n^2} - \frac{1}{n^2} - \frac{(-1)^n}{n^2} - \cancel{\frac{\pi}{2}\frac{\sin n\pi/2}{n}} + \frac{\cos n\pi/2}{n^2}\right.$$

$$= \frac{1}{2}\left[\frac{2\cos n\pi/2 - 1 - (-1)^n}{n^2}\right]$$

$$\therefore f(x) = \frac{\pi^2}{16} + \sum_{n=1}^{\infty}\frac{1}{2}\left[\frac{2\cos n\pi/2 - 1 - (-1)^n}{n^2}\right]\cos nx.$$

HALF RANGE SERIES:

COSINE SERIES:

The half range cosine series in the interval $(0,l)$ is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{l}x$.

Where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$; $a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi}{l}x\,dx$.

Obtain the Fourier cosine series for $f(x) = \begin{cases} kx & , 0 \le x \le l/2 \\ k(l-x) & , l/2 \le x \le l. \end{cases}$

Soln:

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{l}x$.

Now $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$.

$$= \frac{2}{l}\left\{\int_0^{l/2} kx\,dx + \int_{l/2}^{l} k(l-x)\,dx\right\}$$

$$= \frac{2k}{l}\left\{\left[\frac{x^2}{2}\right]_0^{l/2} + \left[lx - \frac{x^2}{2}\right]_{l/2}^{l}\right\}$$

$$= \frac{2k}{l}\left\{\frac{l^2}{8} + \left[\frac{l^2}{2} - \left(\frac{l^2}{2} - \frac{l^2}{8}\right)\right]\right\}$$

$$= \frac{\cancel{2}k}{\cancel{l}} \cdot \frac{\cancel{2}l^{\cancel{2}}}{\cancel{8}\,2}$$

$$= \frac{kl}{2}.$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi}{l}x\,dx$$

$$= \frac{2}{l}\left\{\int_0^{l/2} kx\cos\frac{n\pi}{l}x\,dx + \int_{l/2}^{l} k(l-x)\cos\frac{n\pi}{l}x\,dx\right\}$$

$$= \frac{2k}{l}\left\{\left[x\left(\frac{\sin\left(\frac{n\pi}{l}\right)x}{\left(\frac{n\pi}{l}\right)}\right) - 1\left(-\frac{\cos\left(\frac{n\pi}{l}\right)x}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^{l/2} + \left[(l-x)\left(\frac{\sin\left(\frac{n\pi}{l}\right)x}{\left(\frac{n\pi}{l}\right)}\right) - (-1)\left(-\frac{\cos\left(\frac{n\pi}{l}\right)x}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_{l/2}^{l}\right\}$$

$$= \frac{2k}{l}\left\{\left[\frac{l}{2}\frac{\cancel{\sin n\pi/2}}{\left(\frac{n\pi}{l}\right)} + \frac{\cos n\pi/2}{(n\pi/l)^2} - \frac{\cos 0}{(n\pi/l)^2}\right] + \left[0 - \frac{\cos n\pi}{\left(\frac{n\pi}{l}\right)^2} - \left(\frac{l}{2}\frac{\cancel{\sin n\pi/2}}{\left(\frac{n\pi}{l}\right)} - \frac{\cos n\pi/2}{(n\pi/l)^2}\right)\right]\right\}$$

$$= \frac{2k}{l}\left\{\frac{\cos n\pi/2}{(n\pi/l)^2} - \frac{1}{(n\pi/l)^2} - \frac{(-1)^n}{(n\pi/l)^2} + \frac{\cos n\pi/2}{(n\pi/l)^2}\right\}$$

$$= \frac{2k}{l}\left[\frac{2\cos n\pi/2}{(n\pi/l)^2} - \frac{[1+(-1)^n]}{(n\pi/l)^2}\right]$$

$$= \frac{2kl}{n^2\pi^2}\left[2\cos n\pi/2 - [1+(-1)^n]\right]$$

$$\therefore f(x) = \frac{kl}{4} + \sum_{n=1}^{\infty}\frac{2kl}{n^2\pi^2}\left[2\cos n\pi/2 - (1+(-1)^n)\right]\cos\left(\frac{n\pi}{l}\right)x$$