Harmonic Analysis

Defn: The process of finding the Fourier Series for a function given by numerical values is known as harmonic analysis.

In harmonic analysis the Fourier Coefficients a_0, a_n and b_n of the function $y = f(x)$ in $[0, 2\pi)$ are given by

$$a_0 = 2\,[\text{mean value of } y \text{ in } [0, 2\pi)]$$

$$a_n = 2\,[\text{mean value of } y \cos nx \text{ in } [0, 2\pi)]$$

$$b_n = 2\,[\text{mean value of } y \sin nx \text{ in } [0, 2\pi)]$$

π-form (or) T-form (or) Degree form.

Mean value of $y = \Sigma y / N$

Mean value of $y \cos nx = \Sigma y \cos(nx) / N$

Mean value of $y \sin nx = \Sigma y \sin(nx) / N$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$(0, 2\pi) \Rightarrow l = \pi$

Note:

1) $(a_1 \cos x + b_1 \sin x) \rightarrow$ fundamental or first harmonic

2) $(a_2 \cos 2x + b_2 \sin 2x) \rightarrow$ second harmonic

Type-I π form

1. Find the Fourier Series expansion of period 2π for the function $y = f(x)$ which is defined in $(0, 2\pi)$ by means of the following data, upto Second harmonic.

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
$f(x)$	1.0	1.4	1.9	1.7	1.5	1.2	1.0

repeated value of y

Solution:

Here total number of x's $N = 6$.

$$2l = 2\pi$$

$$\Rightarrow l = \pi.$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

$$= \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x$$

$a_0 = 2(\Sigma y/N)$ $a_n = 2(\Sigma y \cos nx/N)$ $b_n = 2(\Sigma y \sin nx/N)$

x	$y = f(x)$	$y \cos x$	$y \cos 2x$	$y \sin x$	$y \sin 2x$
0	1.0	1	1	0	0
$\pi/3$ 60°	1.4	0.7	−0.7	1.212	1.212
$2\pi/3$ 120°	1.9	−0.95	−0.95	1.645	−1.645
π 180°	1.7	−1.7	1.7	0	0
240° $4\pi/3$	1.5	−0.75	−0.75	−1.299	1.299
300° $5\pi/3$	1.2	0.6	−0.6	−1.039	−1.039
~~2π~~	$\Sigma y = 8.7$	$\Sigma y \cos x = -1.1$	$\Sigma y \cos 2x = -0.3$	$\Sigma y \sin x = 0.5196$	$\Sigma y \sin 2x =$ ~~000~~ -0.1732

$$a_0 = 2\left(\frac{\Sigma y}{N}\right)$$

$$= 2\left(\frac{8.7}{6}\right) = \frac{8.7}{3} = 2.9$$

$$a_1 = 2\left(\frac{\Sigma y \cos x}{N}\right) = 2\left(\frac{-1.1}{6}\right) = -0.37$$

$$a_2 = 2\left(\frac{\Sigma y \cos 2x}{N}\right) = 2\left(\frac{-0.3}{6}\right) = -0.1$$

$$b_1 = 2\left(\frac{\Sigma y \sin x}{N}\right) = 2\left(\frac{0.5196}{6}\right) = 0.17$$

$$b_2 = 2\left(\frac{\Sigma y \sin 2x}{N}\right) = 2\left(\frac{-0.1732}{6}\right) = -0.06$$

$$f(x) = 1.45 - 0.37\cos x - 0.1\cos 2x + 0.17\sin x - 0.06\sin 2x.$$

Type 2: T- form.

1) The following table gives the variations of a Periodic function over a period T.

x	0	$\frac{T}{6}$	$\frac{T}{3}$	$\frac{T}{2}$	$\frac{2T}{3}$	$\frac{5T}{6}$	T
$f(x)$	1.98	1.3	1.05	1.3	-0.88	-0.25	1.98

Find the Fourier Series expansion in $(0,T)$ upto Second harmonic.

Solution: $N = 6$

$T = 2\pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Upto second harmonic

$$= \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x$$

Sub $T = 2\pi$ in the table,

$T = 2\pi$ | $\frac{T}{6} = \frac{2\pi}{6} = \frac{\pi}{3}$ | $\frac{T}{3} = \frac{2\pi}{3}$ | $\frac{T}{2} = \frac{2\pi}{2} = \pi$ | $\frac{2(2\pi)}{3} = \frac{4\pi}{3}$ | $\frac{5(2\pi)}{6} = \frac{5\pi}{3}$

$\frac{T}{6} = \frac{\pi}{3} = \frac{180}{3} = 60^\circ$ | $\frac{T}{3} = \frac{2\pi}{3} = \frac{2(180)}{3} = 120^\circ$ | $\frac{T}{2} = \pi = 180^\circ$ |

$\frac{2T}{3} = \frac{4\pi}{3} = \frac{4(180^\circ)}{3} = 240^\circ$ | $\frac{5T}{6} = \frac{5(2\pi)}{6} = \frac{5(2(180))}{6} = 300^\circ$

x	0	60°	120°	180°	240°	300°
$f(x)$	1.98	1.30	1.05	1.30	−0.88	−0.25

x	$y=f(x)$	$y\cos x$	$y\cos 2x$	$y\sin x$	$y\sin 2x$
0	1.98	1.98	1.98	0	0
0°	1.30	0.65	−0.65	1.126	1.126
0°	1.05	−0.525	−0.525	0.909	−0.909
30°	1.30	−1.3	1.3	0	0
40°	−0.88	0.44	0.44	0.762	−0.762
00°	−0.25	−0.125	0.125	0.217	0.217
	$\Sigma y = 4.5$	$\Sigma y\cos x = 1.12$	$\Sigma y\cos 2x = 2.67$	$\Sigma y\sin x = 3.014$	$\Sigma y\sin 2x = -0.328$

$$a_0 = 2\left(\frac{\Sigma y}{N}\right) = 2\left(\frac{4.5}{6}\right) = 1.5$$

$$a_1 = 2\left(\frac{\Sigma y\cos x}{N}\right) = 2\left(\frac{1.12}{6}\right) = 0.373$$

$$a_2 = 2\left(\frac{\Sigma y\cos 2x}{N}\right) = 2\left(\frac{2.67}{6}\right) = 0.89$$

$$b_1 = 2\left(\frac{\Sigma y\sin x}{N}\right) = 2\left(\frac{3.014}{6}\right) = 1.005$$

$$b_2 = 2\left(\frac{\Sigma y\sin 2x}{N}\right) = 2\left(\frac{-0.328}{6}\right) = -0.109$$

$$f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + b_1\sin x + b_2\sin 2x$$

$$= 0.75 + 0.373\cos x + 0.89\cos 2x + 1.005\sin x - 0.109\sin 2x.$$

3. Find the Fourier Series as far as the se... harmonic to represent the function given in the following data. [l-form]

x	0	1	2	3	4	5
$f(x)$	9	18	24	28	26	20

Solution:

Here the length of the interval is 6

$2l = 6$

$\therefore l = 3$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{3}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{3}\right)$$

$$f(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{3}\right) + a_2 \cos\left(\frac{2\pi x}{3}\right) + b_1 \sin\left(\frac{\pi x}{3}\right) + b_2 \sin\left(\frac{2\pi x}{3}\right)$$

x	$y=f(x)$	$y\cos\left(\frac{\pi x}{3}\right)$	$y\cos\left(\frac{2\pi x}{3}\right)$	$y\sin\left(\frac{\pi x}{3}\right)$	$y\sin$
0	9	9	9	0	0
1	18	9	-9	15.7	15.6
2	24	-12	-24	20.9	0
3	28	-28	28	0	0
4	26	-13	-13	-22.6	22.6
5	20	10	-10	-17.4	-17.4
	$\Sigma y = 125$	$\Sigma y\cos\left(\frac{\pi x}{3}\right) = -25$	$\Sigma y\cos\left(\frac{2\pi x}{3}\right) = -19$	$\Sigma y\sin\left(\frac{\pi x}{3}\right) = -3.4$	$\Sigma y\sin\left(\frac{2\pi x}{3}\right) = 20.8$

$$a_0 = 2\left(\frac{\Sigma y}{N}\right) = 2\left(\frac{125}{6}\right) = 41.66$$

$$a_1 = 2\frac{\left(\Sigma y\cos\left(\frac{\pi x}{3}\right)\right)}{N} = 2\left(\frac{-25}{6}\right) = -8.33$$

$$a_2 = 2\left(\frac{\Sigma y \cos\left(\frac{2\pi x}{3}\right)}{N}\right) = 2\left(\frac{-19}{6}\right) = -6.33$$

$$b_1 = 2\left(\frac{\Sigma y \sin\left(\frac{\pi x}{3}\right)}{N}\right) = 2\left(\frac{-3.4}{6}\right) = -1.13$$

$$b_2 = 2\left(\frac{\Sigma y \sin\left(\frac{2\pi x}{3}\right)}{N}\right) = 2\left(\frac{20.8}{6}\right) = 0.009$$

$$f(x) = 20.83 - 8.33\cos\left(\frac{\pi x}{3}\right) - 6.33\cos\left(\frac{2\pi x}{3}\right) - 1.13\sin\left(\frac{\pi x}{3}\right) + 0.009\sin\left(\frac{2\pi x}{3}\right).$$