1) Expand $f(x) = x - x^2$ as a Fourier Series in $-L < x < L$.

Soln:

Step 1: Fourier Series of $f(x)$ $(-L, L)$

$c = -L$ and $c + 2l = +L$

$-L + 2l = L$

$2l = 2L$

$\boxed{l = L}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

Step 2: To find a_0

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx = \frac{1}{L}\int_{-L}^{L}(x - x^2)\,dx$$

$$= \frac{1}{L}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-L}^{L} = \frac{1}{L}\left[\frac{L^2}{2} - \frac{L^3}{3} - \frac{L^2}{2} - \left(+\frac{L^3}{3}\right)\right]$$

$$= \frac{1}{L}\left[\frac{-2L^3}{3}\right] = \frac{-2L^2}{3} \qquad \boxed{a_0 = \frac{-2L^2}{3}}$$

Step 3: To find a_n

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{L}\int_{-L}^{L}(x - x^2)\cos\left(\frac{n\pi x}{L}\right)dx$$

By Bernoulli's formula

$u = x - x^2$

$u' = 1 - 2x$

$u'' = 0 - 2$

$u''' = 0$

$dv = \cos\left(\frac{n\pi x}{L}\right)dx$

$v = \sin\left(\frac{n\pi x}{L}\right)\cdot\left(\frac{1}{(n\pi/L)}\right) = \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)$

$v_1 = \left(\frac{L}{n\pi}\right)\left(\frac{1}{\left(\frac{n\pi}{L}\right)}\left(-\cos\left(\frac{n\pi x}{L}\right)\right)\right) = \frac{-L^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right)$

$v_2 = \frac{-L^2}{n^2\pi^2}\left(\frac{1}{\left(\frac{n\pi}{L}\right)}\sin\left(\frac{n\pi x}{L}\right)\right)$

$v_2 = \frac{-L^3}{n^3\pi^3}\sin\left(\frac{n\pi x}{L}\right)$

$$a_n = \frac{1}{L}\left[(x - x^2)\left(\frac{L}{n\pi}\right)\sin\left(\frac{n\pi x}{L}\right) - (1 - 2x)\left(\frac{-L^2}{n^2\pi^2}\right)\cos\left(\frac{n\pi x}{L}\right) + (-2)\left(\frac{-L^3}{n^3\pi^3}\right)\sin\left(\frac{n\pi x}{L}\right)\right]_{-L}^{L}$$

$$a_n = \frac{1}{L}\Bigg[(L - L^2)\left(\frac{L}{n\pi}\right)\sin(n\pi) - (1 - 2L)\left(\frac{-L^2}{n^2\pi^2}\right)\cos(n\pi) + \frac{2L^3}{n^3\pi^3}\sin n\pi$$
$$- \left[(-L - L^2)\left(\frac{L}{n\pi}\right)\sin(-n\pi) - (1 + 2L)\left(\frac{-L^2}{n^2\pi^2}\right)\cos(-n\pi) + \frac{2L^3}{n^3\pi^3}\sin(-n\pi)\right]$$

$$= \frac{1}{L}\left[\frac{L^2(1-2L)}{n^2\pi^2}\cos n\pi - \frac{L^2(1+2L)}{n^2\pi^2}\cos n\pi\right]$$

$$= \frac{1}{L} \times \frac{L^2}{n^2\pi^2}\left[\cos n\pi - 2L\cos n\pi - 1\cos n\pi - 2L\cos n\pi\right]$$

$$= \frac{L}{n^2\pi^2}\left[-4L\cos n\pi\right] = \frac{-4L^2}{n^2\pi^2}(-1)^n$$

$$\boxed{a_n = \frac{-4L^2}{n^2\pi^2}(-1)^n}$$

Step 3: To find b_n

$$b_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{1}{L}\int_{-L}^{L}(x-x^2)\sin\left(\frac{n\pi x}{L}\right)dx$$

$u = (x-x^2)$ $\quad dv = \sin\left(\frac{n\pi x}{L}\right)dx$

$u' = 1-2x$ $\quad v = \frac{-L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)$

$u'' = 0-2$ $\quad v_1 = \frac{-L^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right)$

$u''' = 0$ $\quad v_2 = \frac{+L^3}{n^3\pi^3}\cos\left(\frac{n\pi x}{L}\right)$

$$b_n = \frac{1}{L}\left[(x-x^2)\left(\frac{-L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right) - (1-2x)\left(\frac{-L^2}{n^2\pi^2}\right)\sin\left(\frac{n\pi x}{L}\right) - 2\left(\frac{L^3}{n^3\pi^3}\right)\cos\left(\frac{n\pi x}{L}\right)\right]_{-L}^{L}$$

$$= \frac{1}{L}\left[(L-L^2)\left(\frac{-L}{n\pi}\right)\cos(n\pi) - (1-2L)\left(\frac{-L^2}{n^2\pi^2}\right)\sin(n\pi) - 2\left(\frac{L^3}{n^3\pi^3}\right)\cos(n\pi)\right] - \left[(+L+L^2)\left(\frac{+L}{n\pi}\right)\cos(n\pi) - (1+2L)\left(\frac{-L^2}{n^2\pi^2}\right)\sin(n\pi) - 2\left(\frac{L^3}{n^3\pi^3}\right)\cos(n\pi)\right]$$

$$= \frac{1}{L}\left\{\left[\frac{-L(L-L^2)}{n\pi}\cos(n\pi) - \frac{2L^3}{n^3\pi^3}\cos(n\pi)\right] + \left[\frac{(+L+L^2)\cos(n\pi)}{n\pi}\right] - \left[\frac{-L^2}{n\pi}\cos n\pi \mp \frac{L^3}{n\pi} + \frac{2L^3}{n^3\pi^3}\cos(n\pi)\right]\right\}$$

$$= \left[\frac{-(L-L^2)}{n\pi}(-1)^n - \frac{(L+L^2)}{n\pi}(-1)^n \right]$$

$$= \frac{(-1)^n}{n\pi}\left[-L + L^2 - L - L^2\right] = \frac{(-1)^n}{n\pi}(-2L)$$

$$\boxed{b_n = \frac{-2L}{n\pi}(-1)^n}$$

Step 4: Fourier Series of $f(x)$

$$f(x) = \frac{\left(-2L^2/3\right)}{2} + \sum_{n=1}^{\infty} \frac{-4L^2}{n^2\pi^2}(-1)^n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} \frac{-2L}{n\pi}(-1)^n \sin\left(\frac{\pi x n}{L}\right)$$

$$\therefore f(x) = -\frac{L^2}{3} + \sum_{n=1}^{\infty} \frac{4L^2}{n^2\pi^2}(-1)^{n+1}\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} \frac{2L}{n\pi}(-1)^{n+1}\sin\left(\frac{n\pi x}{L}\right)$$