2.4

LAGRANGE'S LINEAR EQUATION:

The equation of the form,

$$\boxed{Pp + Qq = R}$$

is known as Lagrange's Linear equation where P, Q & R are functions of x, y & z.

To solve this equation, it is enough to solve the auxiliary equations,

$$\boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}}$$

The solution of the auxiliary equation of the form $u(x,y,z) = c_1$ & $v(x,y,z) = c_2$, then the solution of the given Lagrange's equation is $\phi(u,v) = 0$.

To solve the A.E we have two methods.

(1) Method of grouping

(2) Method of multipliers.

Method of grouping

① Solve: $px + qy = z$

The given equation is of the form

$$Pp + Qq = R$$

where $P = x$, $Q = y$, $R = z$

The auxiliary equation is,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

i.e., $$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

Take $\frac{dx}{x} = \frac{dy}{y}$

Integrating,

$$\int \frac{dx}{x} = \int \frac{dy}{y}$$

$$\log x = \log y + \log c_1$$

$$\log x - \log y = \log c_1$$

$$\log\left(\frac{x}{y}\right) = \log c_1$$

$$\boxed{\frac{x}{y} = c_1}$$

Take $\frac{dy}{y} = \frac{dz}{z}$

Integrating

$$\int \frac{dy}{y} = \int \frac{dz}{z}$$

$$\log y = \log z + \log c_2$$

$$\log y - \log z = \log c_2$$

$$\log\left(\frac{y}{z}\right) = \log c_2$$

$$\boxed{\frac{y}{z} = c_2}$$

The general solution is $\varphi(c_1, c_2) = 0$

$$\Rightarrow \varphi\left(\frac{x}{y}, \frac{y}{z}\right) = 0.$$

(2) Solve : $pyz + qzx = xy$

Soln: The given equation is of the form.

$Pp + Qq = R$

Where $P = yz$, $Q = zx$, $R = xy$

The subsidiary equation is,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$$

Take $\frac{dx}{y\cancel{z}} = \frac{dy}{\cancel{z}x}$

$$x\,dx = y\,dy$$

$$\int x\,dx = \int y\,dy$$

$$\frac{x^2}{2} = \frac{y^2}{2} + \frac{c_1}{2}$$

$$\frac{x^2}{2} - \frac{y^2}{2} = \frac{c_1}{2}$$

$$\boxed{x^2 - y^2 = c_1}$$

$\frac{dy}{z\cancel{x}} = \frac{dz}{\cancel{x}y}$

$$y\,dy = z\,dz$$

$$\int y\,dy = \int z\,dz$$

$$\frac{y^2}{2} = \frac{z^2}{2} + \frac{c_2}{2}$$

$$\frac{y^2}{2} - \frac{z^2}{2} = \frac{c_2}{2}$$

$$\boxed{y^2 - z^2 = c_2}$$

The general solution is,

$$\phi(c_1, c_2) = 0$$

$$\phi(x^2 - y^2, y^2 - z^2) = 0$$

Method of multipliers

(1) Solve : $x(y-z)p + y(z-x)q = z(x-y)$

Soln: The given equation is of the form,

$$Pp + Qq = R$$

where $P = x(y-z)$, $Q = y(z-x)$, $R = z(x-y)$

The subsidiary equation is,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} \rightarrow (1)$$

Using the multipliers 1, 1, 1 we get

$$(1) = \frac{dx+dy+dz}{x(y-z)+y(z-x)+z(x-y)} = \frac{dx+dy+dz}{\cancel{xy}-\cancel{xz}+\cancel{yz}-\cancel{yx}+\cancel{zx}-\cancel{zy}}$$

$$(1) = \frac{dx+dy+dz}{0}$$

$$dx+dy+dz=0$$

Integrating, $\int dx + \int dy + \int dz = 0$

$$\boxed{x+y+z=c_1}$$

Using the multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ we get

$$(1) = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\frac{1}{\cancel{x}}\cdot\cancel{x}(y-z) + \frac{1}{\cancel{y}}\cdot\cancel{y}(z-x) + \frac{1}{\cancel{z}}\cdot\cancel{z}(x-y)}$$

Scanned by CamScanner

$$(1) = \frac{\frac{1}{x}\,dx + \frac{1}{y}\,dy + \frac{1}{z}\,dz}{0}$$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integrating, $\int \frac{dx}{x} + \int \frac{dy}{y} + \int \frac{dz}{z} = 0$

$$\log x + \log y + \log z = \log c_2$$

$$\log (xyz) = \log c_2$$

$$\boxed{xyz = c_2}$$

The general solution is,

$$\varphi(c_1, c_2) = 0$$

$$\boxed{\varphi(x+y+z,\ xyz) = 0}$$

Scanned by CamScanner

(5) Solve : $x(z^2-y^2)p+y(x^2-z^2)q = z(y^2-x^2)$

Solution:

The given equation is of the form, $Pp+Qq=R$

Here $P = xz^2-xy^2$, $Q = x^2y-z^2y$, $R = y^2z-zx^2$

The A.E is $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

i.e., $\dfrac{dx}{xz^2-xy^2} = \dfrac{dy}{x^2y-z^2y} = \dfrac{dz}{y^2z-zx^2}$

(21)

Using the multipliers x, y, z we get,

$$\text{Each ratio} = \frac{x\,dx + y\,dy + z\,dz}{x^2z^2 - x^2y^2 + x^2y^2 - y^2z^2 + z^2y^2 - z^2x^2}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

$\therefore\ x\,dy + y\,dy + z\,dz = 0.$

Integrating, $\int x\,dx + \int y\,dy + \int z\,dz = 0$

$$x^2 + y^2 + z^2 = c_1$$

$$\boxed{u(x, y, z) = x^2 + y^2 + z^2}$$

Using the multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ we get,

$$\text{Each ratio} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{z^2 - y^2 + x^2 - z^2 + y^2 - z^2}$$

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\Rightarrow \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0.$$

Integrating,

$$\int \frac{dx}{x} + \int \frac{dy}{y} + \int \frac{dz}{z} = 0$$

$$\log x + \log y + \log z = \log c_2$$

$$\log(xyz) = \log c_2$$

$$xyz = c_2$$

$$\therefore \boxed{v(x, y, z) = xyz}$$

$\therefore$ The solution is $\phi(u, v) = 0$

i.e., $\boxed{\phi(x^2 + y^2 + z^2,\ xyz) = 0}$

Scanned by CamScanner