(15)

Linear and Non-linear PDE:

A PDE is said to be 'linear', if p and q have the first degree otherwise is called non-linear PDE.

Ex: 1) $x^2 \frac{\partial^2 z}{\partial x^2} + x \frac{\partial z}{\partial y} + y \frac{\partial z}{\partial y} + z = e^{x+y}$ is linear.

2) $\frac{\partial^2 z}{\partial x^2} + z\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) = 0$ & $\frac{\partial^2 z}{\partial x^2} + z^2 = \sin(x+y)$ are non-linear.

Solutions of PDE:

Defn: A solution of a PDE is a relation between the dependent and independent variables which satisfies the PDE.

Types of Solutions:

Complete integral: A solution which contains as many arbitrary constants as there are independent variables is called a complete integral or complete solution.

Particular integral: A solution got by giving particular values to the arbitrary constants in a complete integral is called a particular integral.

To find the singular integral:

Let $f(x, y, z, p, q) = 0 \rightarrow (1)$ be the PDE, whose complete integral is,

$$\phi(x, y, z, a, b) = 0 \rightarrow (2)$$

Diff (2) partially w.r.t 'a' & 'b' then equal to zero, we get,

$$\frac{\partial \phi}{\partial a} = 0 \rightarrow (3), \quad \frac{\partial \phi}{\partial b} = 0 \rightarrow (4)$$

Scanned by CamScanner

Eliminate a and b from the equations (2), (3) & (4) when it exists, is called the singular integral.

To find the general integral:

In the complete integral (2), we assume that $b = f(a)$.

Then (2) becomes,

$$\varphi(x, y, z, a, f(a)) = 0 \rightarrow (5)$$

Diff (5) partially w.r.t 'a',

$$\frac{\partial \varphi}{\partial a} + \frac{\partial \varphi}{\partial b} f'(a) = 0 \rightarrow (6)$$

Eliminate 'a' between these two eqns (3) & (4) if it exists, is called the G.I of (1).

(11)

TYPE I : Equation of the form $F(p, q) = 0 \rightarrow (1)$

Step 1 : Let us assume that $z = ax + by + c \rightarrow (2)$ be the solution of (1).

Step 2 : Put $p = a$ & $q = b$ in (1) we get a relation connecting a & b.

Step 3 : We can find 'b' in terms of 'a' and substitute for b in (2) we get the complete solution.

Step 4 : No Singular solution

Step 5 : Find the General Solution

Problems :

(1) Solve : $p + q = pq$.

Soln : Given : $p + q = pq \rightarrow (1)$

Let us assume that $z = ax + by + c \rightarrow (2)$ be the solution of (1).

Put $p = a$ & $q = b$ in (1), we get

$$a + b = ab$$

$$a = b(a-1) \Rightarrow \boxed{b = \frac{a}{a-1}}$$

Subs 'b' in (2), we get

$$\boxed{z = ax + \left(\frac{a}{a-1}\right)y + c} \rightarrow (3)$$

which is the complete solution of (1).

To find singular soln :

Diff (3) partially w.r.t 'c', we get

$0 = 1$ which is not possible.

$\therefore$ There is no singular solution.

To find General soln :

Put $c = \varphi(a)$ in (3), we get

Scanned by CamScanner

$$z = ax + \left(\frac{a}{a-1}\right) y + \phi(a) \longrightarrow (4)$$

Diff (4) partially w.r.t 'a',

$$0 = x - \frac{y}{(a-1)^2} + \phi'(a) \longrightarrow (5)$$

Eliminate 'a' from (4) & (5), we get the general soln.

(2) Solve : $\sqrt{p} + \sqrt{q} = 1$

Soln: Given : $\sqrt{p} + \sqrt{q} = 1 \longrightarrow (1)$

Let us assume that $z = ax + by + c \longrightarrow (2)$ be the solution of (1).

Put $p = a$ & $q = b$ in (1), we get

$$\sqrt{a} + \sqrt{b} = 1$$

$$\sqrt{b} = 1 - \sqrt{a}$$

$$\boxed{b = \pm (1-\sqrt{a})^2}$$

Subs 'b' in (2), we get

$$z = ax \pm (1-\sqrt{a})^2 y + c \longrightarrow (3)$$

Which is the complete solution of (1).

To find the singular solution :

Diff (3) partially w.r.t 'c',

$$0 = 1$$

Which is not possible. ∴ There is no singular solution.

To find the general solution :

Put $c = \phi(a)$ in (3), we get

$$z = ax + (1-\sqrt{a})^2 y + \phi(a) \longrightarrow (4)$$

Diff (4) partially w.r.t 'a',

$$0 = x + 2(1-\sqrt{a}) \left(\frac{-1}{2\sqrt{a}}\right) y + \phi'(a) \longrightarrow (5)$$

Eliminating 'a' from (4) & (5) we get the general solution.

Scanned by CamScanner

12

(3) Solve : $p^2+q^2 = 4pq$.

Soln :

Given : $p^2+q^2 = 4pq \rightarrow (1)$

Let us assume that $z = ax+by+c \rightarrow (2)$ be the Solution of (1), put $p = a$ & $q = b$ in (1), we get

$$a^2+b^2 = 4ab$$

$$\Rightarrow b^2-4ab+a^2 = 0$$

$$\Rightarrow b = \frac{4a \pm \sqrt{16a^2-4a^2}}{2} = \frac{4a \pm \sqrt{12a^2}}{2} = \frac{4a \pm 2a\sqrt{3}}{2}$$

$$b = a(2 \pm \sqrt{3})$$

subs the value of 'b' in (2), we get

$$\boxed{z = ax + a(2 \pm \sqrt{3}) + c} \rightarrow (3)$$

which is the complete solution.

To find the Singular Solution :

Diff (3) partially w.r.t 'c', we get

$$0 = 1$$

which is not possible.

There is no singular solution.

To find the General Solution :

Put $c = \phi(a)$ in (3), we get

$$z = ax + a(2 \pm \sqrt{3}) + \phi(a) \rightarrow (4)$$

Diff (4) partially w.r.t 'a', we get.

$$0 = x + (2 \pm \sqrt{3}) + \phi'(a) \rightarrow (5)$$

Eliminate 'a' from (4) & (5) we get the General Solution.

13

Type 2 : Equation of the form $z = px + qy + f(p,q)$ (Clairaut's form) :

Step 1 : Equation of the form put $p = a$ & $q = b$ we get the complete solution as $z = ax + by + f(a,b)$

Step 2 : Find the Singular Solution

Step 3 : Find the General Solution.

Problems :

① Solve : $z = px + qy - 2\sqrt{pq}$

Soln :

Put $p = a$ & $q = b$ then the complete solution is given by,

$$z = ax + by - 2\sqrt{ab} \rightarrow (1)$$

To find the singular solution :

Diff (1) partially w.r.t 'a',

$$0 = x + 0 - 2\sqrt{b}\left(\frac{1}{2\sqrt{a}}\right)$$

$$x = \frac{\sqrt{b}}{\sqrt{a}} \rightarrow (2)$$

Diff (1) partially w.r.t 'b', we get,

$$0 = y - 2\sqrt{a}\left(\frac{1}{2\sqrt{b}}\right)$$

$$\Rightarrow y = \frac{\sqrt{a}}{\sqrt{b}} \rightarrow (3)$$

From (2) & (3), we get

$$xy = \frac{\sqrt{b}}{\sqrt{a}} \cdot \frac{\sqrt{a}}{\sqrt{b}} = 1$$

$\therefore \boxed{xy = 1}$ which is the singular solution.

(2) Solve : $z = px + qy + p^2 + q^2$.

Soln: Equation of the form : $z = px + qy + f(p, q)$.

Hence the complete solution is,

$$z = ax + by + a^2 + b^2 \longrightarrow (1)$$

To find the singular solution:

Diff (1) partially w.r.t 'a',

$$0 = x + 2a \Rightarrow \boxed{a = \frac{-x}{2}} \longrightarrow (2)$$

Diff (1) partially w.r.t 'b',

$$0 = y + 2b \Rightarrow \boxed{b = -\frac{y}{2}} \longrightarrow (3)$$

subs 'a' & 'b' in (1), we get,

$$z = -\frac{x^2}{2} - \frac{y^2}{2} + \frac{x^2}{4} + \frac{y^2}{4}$$

$$z = -\frac{x^2}{4} - \frac{y^2}{4}$$

$\boxed{4z + x^2 + y^2 = 0}$ which is the singular solution.

Scanned by CamScanner

(15)

(4) Solve : $z = px + qy + \sqrt{1+p^2+q^2}$

Soln:

Eqn of the form : $z = px + qy + f(p, q)$

Hence the complete solution is,

$$z = ax + by + \sqrt{1+a^2+b^2} \rightarrow (1)$$

To find the singular solution:

Diff (1) partially w.r.t 'a',

$$0 = x + \frac{1}{2}(1+a^2+b^2)^{-1/2}(2a)$$

$$x = \frac{-a}{\sqrt{1+a^2+b^2}} \Rightarrow \frac{x}{a} = \frac{-1}{\sqrt{1+a^2+b^2}} \rightarrow (2)$$

Diff (1) partially w.r.t 'b',

$$0 = y + \frac{1}{2}(1+a^2+b^2)^{-1/2} \cdot 2b$$

$$y = \frac{-b}{\sqrt{1+a^2+b^2}} \Rightarrow \frac{y}{b} = \frac{-1}{\sqrt{1+a^2+b^2}} \rightarrow (3)$$

From (2) & (3), $\frac{x}{a} = \frac{y}{b} = \frac{1}{k}$ (say)

Take, $\frac{x}{a} = \frac{1}{k} \Rightarrow a = xk$

Take, $\frac{y}{b} = \frac{1}{k} \Rightarrow b = yk$

From (2), $\frac{1}{k} = \frac{-1}{\sqrt{1+x^2k^2+y^2k^2}}$

$$k = -\sqrt{1+x^2k^2+y^2k^2}$$

$$k^2 = 1+x^2k^2+y^2k^2$$

$$(1-x^2-y^2)k^2 = 1$$

$$k^2 = \frac{1}{1-x^2-y^2}$$

$$k = \frac{1}{\sqrt{1-x^2-y^2}}$$

Scanned by CamScanner

Type 3 : Equation of the form $F(z, p, q) = 0 \rightarrow (1)$

where $u = x + ay$

subs the values of a & b in (1),

$$z = x^2 k^2 + y^2 k + \sqrt{1 + x^2 k^2 + y^2 k^2}$$

$$= x^2 k + y^2 k - k$$

$$= k(x^2 + y^2 - 1)$$

$$z = \frac{1}{\sqrt{1 - x^2 - y^2}} (-1)(1 - x^2 - y^2)$$

$$z^2 = \frac{(1 - x^2 - y^2)^2}{1 - x^2 - y^2}$$

$$z^2 = 1 - x^2 - y^2$$

$\boxed{z^2 + x^2 + y^2 = 1}$ which is the singular solution.