UNIT – II

FOURIER TRANSFORMS

Fourier transforms :

Fourier transform of $f(x)$ is defined as,

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

Inverse Fourier transforms :

Inverse Fourier transform ~~of~~ is defined as,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F[f(x)]\, e^{-isx}\, ds$$

* Fourier transform and Inverse Fourier transform are jointly called as fourier transform pair.

Parseval's Identity :

If $F(s)$ is a Fourier transform of $f(x)$, then

$$\int_{-\infty}^{\infty} [F(s)]^2\, ds = \int_{-\infty}^{\infty} [f(x)]^2\, dx$$

Note :

* $e^{i\theta} = \cos\theta + i\sin\theta$
* $e^{-i\theta} = \cos\theta - i\sin\theta$
* $\int_{-\infty}^{\infty} f(x)\, dx = 0$, if $f(x)$ is odd.
* $\int_{-\infty}^{\infty} f(x)\, dx = 2\int_{0}^{\infty} f(x)\, dx$, if $f(x)$ is even.

Problems:

(1) Find the Fourier transform of

$$f(x) = \begin{cases} a-|x|, & |x| < a \\ 0, & |x| > a \end{cases}$$ and hence prove

(i) $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$ (ii) $\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$

Solution:

The Fourier transform of $f(x)$ is,

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} [a-|x|]\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} [a-|x|]\, (\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-a}^{a} [a-|x|] \cos sx\, dx + \int_{-a}^{a} [a-|x|]\, i \sin sx\, dx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ 2\int_0^a (a-x) \cos sx\, dx + 0 \right\}$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^a (a-x) \cos sx\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \left\{ \cancel{(a-x)\frac{\sin sx}{s}}^{\,0} - \frac{\cos sx}{s^2} \right\}_0^a$$

$u = a - x$	$v = \cos sx$
$u' = -1$	$v_1 = \frac{\sin sx}{s}$
$u'' = 0$	$v_2 = \frac{-\cos sx}{s^2}$

$$= \frac{-2}{\sqrt{2\pi}\, s^2} [\cos sa - 1]$$

$$\boxed{F(s) = \frac{2}{\sqrt{2\pi}\, s^2} [1 - \cos sa]}$$

(i) Using inverse Fourier transform,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{2}{\sqrt{2\pi}\, s^2} [1 - \cos sa]\, e^{-isx}\, ds$$

$$= \frac{\cancel{2}}{\cancel{2}\pi} \int_{-\infty}^{\infty} \frac{1-\cos sa}{s^2}\, e^{-isx}\, ds$$

$$f(x) = \frac{1}{\pi} \cdot 2 \int_0^{\infty} \frac{1-\cos sa}{s^2}\, e^{-isx}\, ds$$

$$a - |x| = \frac{2}{\pi} \int_0^{\infty} \frac{1-\cos sa}{s^2}\, e^{-isx}\, ds$$

Put $a = 2,\ x = 0,\ s = t \Rightarrow ds = dt$ $\quad \boxed{\frac{1-\cos 2t}{2} = \sin^2 t}$

$$2 - 0 = \frac{2}{\pi} \int_0^{\infty} \frac{2\sin^2 t}{t^2}\, dt$$

$$2 = \frac{4}{\pi} \int_0^\infty \frac{\sin^2 t}{t^2}\, dt$$

$$\therefore \int_0^\infty \frac{\sin^2 t}{t^2}\, dt = \frac{\pi}{2}$$

(ii) Using Parseval's identity,

$$\int_{-\infty}^{\infty} [F(s)]^2\, ds = \int_{-\infty}^{\infty} [f(x)]^2\, dx$$

$$\int_{-\infty}^{\infty} \left[\frac{2}{\sqrt{2\pi}\, s^2} [1 - \cos sa]\right]^2 ds = \int_{-a}^{a} (a - |x|)^2\, dx$$

$$\frac{\cancel{4}^{\,2}}{\cancel{2}\pi} \int_{-\infty}^{\infty} \frac{(1-\cos sa)^2}{s^4}\, ds = 2\int_0^a (a-x)^2\, dx$$

$$\frac{2}{\pi} \times 2 \int_0^\infty \frac{(1-\cos sa)^2}{s^4}\, ds = 2\left[\frac{(a-x)^3}{-3}\right]_0^a$$

$$\frac{4}{\pi} \int_0^\infty \frac{(1-\cos sa)^2}{s^4}\, ds = -\frac{2}{3}\left[0 - a^3\right]$$

$$\frac{4}{\pi} \int_0^\infty \frac{(1-\cos sa)^2}{s^4}\, ds = \frac{2a^3}{3}$$

Put $s = t$, $a = 2$ then

$$\frac{4}{\pi} \int_0^\infty \frac{(1-\cos 2t)^2}{t^4}\, dt = \frac{2}{3} \times 2^3$$

$$\frac{4}{\pi} \int_0^\infty \frac{(2\sin^2 t)^2}{t^4}\, dt = \frac{16}{3}$$

$$\frac{16}{\pi}\int_0^\infty \frac{\sin^4 t}{t^4}\,dt = \frac{16}{3}$$

$$\int_0^\infty \frac{\sin^4 t}{t^4}\,dt = \frac{\cancel{16}}{3} \times \frac{\pi}{\cancel{16}}$$

$$\therefore \boxed{\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}}$$