(4) Find the Fourier transforms of

$$f(x) = \begin{cases} 1 & \text{in } |x| < a \\ 0 & \text{in } |x| > a > 0 \end{cases}$$

Hence deduce that (i) $\int_0^\infty \left(\frac{\sin t}{t}\right) dt = \frac{\pi}{2}$ (ii) $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$

Solution:

Given: $f(x) = \begin{cases} 1 & , -a < x < a \\ 0 & , -\infty < x < -a \ \& \ 0 < x < \infty \end{cases}$

(i) $$F(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} 1 \cdot e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot 2 \int_0^a \cancel{e^{isx}} \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s}\right]_0^a$$

$$\boxed{F(f(x)) = \sqrt{\frac{2}{\pi}}\, \frac{\sin sa}{s}}$$

(ii) Using inverse Fourier transforms,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left(\frac{\sin as}{s}\right) e^{-isx}\, ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin as}{s}\right)(\cos sx - i \sin sx)\, ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin sa}{s}\right) \cos sx\, ds$$

$$f(x) = \frac{1}{\pi} \cdot 2 \int_0^{\infty} \left(\frac{\sin as}{s}\right) \cos sx\, ds$$

$$\Rightarrow \int_0^{\infty} \left(\frac{\sin as}{s}\right) \cos sx\, ds = \frac{\pi}{2} f(x)$$

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Putting $a = 1$,

$$\int_0^\infty \left(\frac{\sin s}{s}\right) \cos sx \, ds = \frac{\pi}{2} f(x)$$

Putting $x = 0$, $f(x) = f(0) = 1$

$$\int_0^\infty \left(\frac{\sin s}{s}\right) ds = \frac{\pi}{2}$$

Replace 's' by 't', we get

$$\int_0^\infty \left(\frac{\sin t}{t}\right) dt = \frac{\pi}{2}$$

(iii) Using Parseval's identity,

$$\int_{-\infty}^\infty |F(s)|^2 \, ds = \int_{-\infty}^\infty |f(x)|^2 \, dx$$

$$\int_{-\infty}^\infty \left(\frac{\sin s}{s}\right)^2 ds = \int_{-1}^1 dx$$

$$2 \times \frac{2}{\pi} \int_0^\infty \left(\frac{\sin s}{s}\right)^2 ds = (x)_{-1}^1$$

$$= 2$$

$$\int_0^\infty \left(\frac{\sin s}{s}\right)^2 ds = \frac{\pi}{2}$$

Replace 's' by 't', we get

$$\boxed{\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}}$$

(5) Evaluate $\displaystyle\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$ using transforms.

Solution:

Let $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$, $a, b > 0$

We know that,

$$F_c(e^{-ax}) = \sqrt{\frac{2}{\pi}}\left(\frac{a}{s^2+a^2}\right)$$

$$G_c(e^{-bx}) = \sqrt{\frac{2}{\pi}}\left(\frac{b}{s^2+b^2}\right)$$

By property,

$$\int_0^\infty f(x)\,g(x)\,dx = \int_0^\infty F_c(f(x))\,G_c(g(x))\,ds$$

$$\int_0^\infty e^{-ax} e^{-bx}\,dx = \int_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{a}{s^2+a^2}\right)\sqrt{\frac{2}{\pi}}\left(\frac{b}{s^2+a^2}\right) ds$$

$$\int_0^\infty e^{-(a+b)x}\,dx = \frac{2ab}{\pi}\int_0^\infty \frac{ds}{(s^2+a^2)(s^2+b^2)}$$

$$\therefore \int_0^\infty \frac{ds}{(s^2+a^2)(s^2+b^2)} = \frac{1}{a+b}\cdot\frac{\pi}{2ab} = \frac{\pi}{2ab(a+b)}$$

Replacing 's' by 'x', we get

$$\boxed{\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}}$$

(6) Evaluate $\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)}$ using Transforms.

Solution:

Let $f(x) = e^{-ax}$, $g(x) = e^{-bx}$

We know that,

$$F_s(e^{-ax}) = \sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right)$$

$$G_s(e^{-bx}) = \sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+b^2}\right)$$

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By property,

$$\int_0^\infty f(x)\, g(x)\, dx = \int_0^\infty F_s(f(x))\, G_s(g(x))\, ds$$

$$\int_0^\infty e^{-(a+b)x}\, dx = \frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}\, ds$$

$$\frac{1}{a+b} = \frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}\, ds$$

$$\int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}\, ds = \frac{\pi}{2(a+b)}$$

Replace 's' by 'x', we get,

$$\boxed{\int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}\, dx = \frac{\pi}{2(a+b)}}$$

(7) Using parseval's identity find (i) $\int_0^\infty \frac{dx}{(a^2+x^2)^2}$

(ii) $\int_0^\infty \frac{x^2}{(a^2+x^2)^2}\, dx$ if $a>0$

Solution :

(i) Let $f(x) = e^{-ax}$

$$F_c(f(x)) = F_c(e^{-ax}) = \sqrt{\frac{2}{\pi}}\left(\frac{a}{s^2+a^2}\right)$$

Parseval's identity for Fourier cosine transforms is,

$$\int_0^\infty |f(x)|^2\, dx = \int_0^\infty |F_c(f(x))|^2\, ds$$

$$\Rightarrow \int_0^\infty (e^{-ax})^2\, dx = \int_0^\infty \left(\sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}\right)^2 ds$$

$$\left(\frac{e^{-2ax}}{-2a}\right)_0^\infty = \frac{2}{\pi}a^2\int_0^\infty \frac{ds}{(s^2+a^2)^2}$$

$$\frac{1}{2a} = \frac{2a^2}{\pi}\int_0^\infty \frac{ds}{(s^2+a^2)^2}$$

$$\int_0^\infty \frac{ds}{(s^2+a^2)^2} = \frac{1}{2a}\cdot\frac{\pi}{2a^2} = \frac{\pi}{4a^3}$$

Replace 's' by 'x', we get,

$$\boxed{\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}}$$

(ii) We know that,

$$F_s(f(x)) = \sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right)$$

Parseval's identity for Fourier sine transforms is,

$$\int_0^\infty |f(x)|^2\,dx = \int_0^\infty |F_s(f(x))|^2\,ds$$

$$\int_0^\infty e^{-2ax}\,dx = \int_0^\infty \left(\sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right)\right)^2 ds$$

$$\frac{1}{2a} = \frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)^2}\,ds$$

$$\int_0^\infty \frac{s^2}{(s^2+a^2)^2}\,ds = \frac{1}{2a}\cdot\frac{\pi}{2} = \frac{\pi}{4a}$$

Replacing 's' by 'x' we get,

$$\boxed{\int_0^\infty \frac{x^2}{(x^2+a^2)^2}\,dx = \frac{\pi}{4a}}$$