SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

DEPARTMENT OF MATHEMATICS

Problems:

1. Find Fourier transform of

$$f(x) = \begin{cases} x, & |x| \le a \quad (-a \le x \le a) \\ 0, & |x| > a \end{cases}$$

Soln:

Fourier transform of $f(x)$ is,

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x\, e^{isx}\, dx.$$

class

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x\,(\cos sx + i \sin sx)\, dx.$$

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$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{a} x \cos sx\, dx + i \int_{-a}^{a} x \sin sx\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[0 + i \cdot 2 \int_{0}^{a} x \sin sx\, dx \right]$$

$$= \frac{2i}{\sqrt{2\pi}} \int_{0}^{a} x \sin sx\, dx$$

$$= i \cdot \frac{2}{\sqrt{2\pi}} \left[\frac{-x}{s} \cos sx + \frac{\sin sx}{s^2} \right]_0^a$$

$$= i \cdot \frac{2}{\sqrt{2\pi}} \left[\frac{-a \cos as}{s} + \frac{\sin sa}{s^2} \right]$$

$$= i \sqrt{\frac{2}{\pi}} \left[\frac{\sin as - as \cos as}{s^2} \right]$$

(2) Find the Fourier transform of

$$f(x) = \begin{cases} x^2, & |x| \le a \\ 0, & |x| > a \end{cases}$$

Soln: Fourier transform of $f(x)$ is,

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx.$$

HW

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{a} x^2 e^{isx} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{a} x^2 (\cos sx + i \sin sx) dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{a} x^2 \cos sx \, dx + i \int_{-a}^{a} x^2 \sin sx \, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[2 \int_{0}^{a} x^2 \cos sx \, dx + 0 \right]$$

$$= \frac{2}{\sqrt{2\pi}} \left[x^2 \left(\frac{\sin sx}{s} \right) - 2x \left(\frac{-\cos sx}{s^2} \right) + 2 \left(\frac{-\sin sx}{s^3} \right) \right]_0^a$$

$$= \frac{2}{\sqrt{2\pi}} \left[a^2 \frac{\sin as}{s} + \frac{2a \cos as}{s^2} - \frac{2 \sin as}{s^3} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a^2 s^2 \sin as + 2as \cos as - 2 \sin as}{s^3} \right]$$

(3) Find F.T of $f(x) = \begin{cases} e^{ikx}, & a < x < b \\ 0, & x < a \ \& \ x > b \end{cases}$

Soln:

$$F(s) = \frac{1}{\sqrt{2\pi} \, i(k+s)} \left[e^{ib(k+s)} - e^{ia(k+s)} \right]$$

(4) Find Fourier transform of $f(x) = \begin{cases} 1, & a<x<b \\ 0, & x<a \text{ \& } x>b \end{cases}$

HW

Soln:

$$F(s) = \frac{1}{\sqrt{2\pi}\, is}\left[e^{ibs} - e^{ias}\right]$$

(5) Find F.T of $f(x) = \begin{cases} 1, & |x|<a \\ 0, & |x|>a \end{cases}$ & hence find (i) $\int_0^\infty \frac{\sin x}{x}\,dx$ (ii) $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt$ (or) $\int_0^\infty \frac{\sin^2 t}{t^2}\,dt$.

Soln:

$$F(s) = F[f(x)] = \frac{2}{\sqrt{2\pi}}\left(\frac{\sin as}{s}\right).$$

(3)

(i) Inverse Fourier transform of $f(x)$ is,

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{2}{\sqrt{2\pi}}\left(\frac{\sin as}{s}\right)(\cos sx - i\sin sx)\, ds.$$

$f(s) = \frac{\sin as}{s}$

$f(-s) = \frac{\sin(-as)}{-s} = +\frac{\sin as}{+s}$

Eve

$$= \frac{\cancel{2} \times 2}{\cancel{2}\pi}\int_0^\infty \frac{\sin as}{s}\cos sx\, ds.$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin as}{s}\cos sx\, ds.$$

$$\therefore \int_0^\infty \left(\frac{\sin as}{s}\right)\cos sx\, ds = \frac{\pi}{2} f(x).$$

Put $x = 0$,

$$\int_0^{\infty} \left(\frac{\sin as}{s}\right) \cos 0 \, ds = \frac{\pi}{2} f(0)$$

$$\int_0^{\infty} \frac{\sin as}{s} \, ds = \frac{\pi}{2} (1)$$

Let $as = x \Rightarrow a.ds = dx$.

$s = x/a$.

$$\therefore \int_0^{\infty} \frac{\sin x}{x/\cancel{a}} \cdot \frac{dx}{\cancel{a}} = \frac{\pi}{2}.$$

$$\therefore \boxed{\int_0^{\infty} \frac{\sin x}{x} \, dx = \frac{\pi}{2}}$$

(ii) Using Parseval's identity,

$$\int_{-\infty}^{\infty} (f(x))^2 \, dx = \int_{-\infty}^{\infty} (F(s))^2 \, ds.$$

$$\int_{-a}^{a} 1^2 \, dx = \int_{-\infty}^{\infty} \left(\frac{2}{\sqrt{2\pi}} \frac{\sin as}{s}\right)^2 ds.$$

$$\cancel{2}a = \frac{\cancel{2}}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin as}{s}\right)^2 ds$$

$$a\pi = \int_{-\infty}^{\infty} \left(\frac{\sin as}{s}\right)^2 ds$$

$$\pi a = 2 \int_0^{\infty} \left(\frac{\sin as}{s}\right)^2 ds$$

$$\int_0^{\infty} \left(\frac{\sin as}{s}\right)^2 ds = \frac{\pi a}{2}$$

(9) $f(x) = \begin{cases} 1-|x|, & |x|<1 \\ 0, & |x|>1 \end{cases}$

HW P.T (i) $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$ (ii) $\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$

Self-Reciprocal

If a transformation of a function $f(x)$ is $F(s)$ then the function $f(x)$ is called self reciprocal.

Example: $f(x) = e^{-x^2/2}$

$$F[f(x)] = F(s) = e^{-s^2/2} = f(s)$$

It is self-reciprocal under Fourier transforms.

Note: $\boxed{\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}}$

If $F[f(x)]$ is $f(s)$, then $f(x)$ is self reciprocal under Fourier transform

Problems based on self-reciprocal:

(1) Find Fourier transform of $e^{-a^2x^2}$ and hence find F.T of $e^{-x^2/2}$

Soln:

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2 - isx)} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[(ax)^2 - isx + \left(\frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} e^{\left(\frac{is}{2a}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-s^2/4a^2} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} dx$$

Put $t = ax - \frac{is}{2a}$

$dt = a\,dx \Rightarrow dx = dt/a$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \frac{e^{-s^2/4a^2}}{a} \int_{-\infty}^{\infty} e^{-t^2} dt$$

$$= \frac{e^{-s^2/4a^2}}{a\sqrt{2}\cancel{\sqrt{\pi}}} \cancel{\sqrt{\pi}}$$

$$F[f(x)] = F(s) = \frac{e^{-s^2/4a^2}}{a\sqrt{2}}$$

Put $s = x$, $a = 1/\sqrt{2}$,

$$F(e^{-x^2/2}) = e^{-s^2/2}$$

(2) Show that $e^{-x^2/2}$ is self reciprocal w.r.t Fourier transform.

Soln:

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty} f(x)\, e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x^2}{2} - isx\right)} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x}{\sqrt{2}} - \frac{is}{\sqrt{2}}\right)^2} e^{-s^2/2} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-s^2/2} \int_{-\infty}^{\infty} e^{-\left(\frac{x}{\sqrt{2}} - \frac{is}{\sqrt{2}}\right)^2} dx$$

Take $t = \frac{x}{\sqrt{2}} - \frac{is}{\sqrt{2}}$

$$dt = \frac{dx}{\sqrt{2}} \Rightarrow dx = \sqrt{2}\, dt.$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} e^{-s^2/2} \int_{-\infty}^{\infty} e^{-t^2} \sqrt{2}\, dt$$

$$= \frac{1}{\sqrt{\pi}} e^{-s^2/2} \int_{-\infty}^{\infty} e^{-t^2} dt$$

$$= \frac{1}{\cancel{\sqrt{\pi}}} e^{-s^2/2} \cancel{\sqrt{\pi}}$$

$$\boxed{F(s) = e^{-s^2/2}}$$

$\therefore e^{-x^2/2}$ is self reciprocal under Fourier transforms.

Convolution:

The convolution of two functions $f(x)$ and $g(x)$ is denoted by,

$$(f*g)(x) = f(x)*g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$$

Convolution theorem:

If $F[f(x)]$ and $F[g(x)]$ are the Fourier transforms of $f(x)$ and $g(x)$ respectively. Then the Fourier transform of convolution of $f(x) \cdot g(x)$ is the product of their Fourier transforms

$$F[f(x) * g(x)] = F(s) \cdot G(s) = F[f(x)] \cdot F[g(x)]$$