Formula:

1. $\int_0^\infty e^{-ax} \cos sx \, dx = \frac{a}{a^2+s^2}$

2. $\int_0^\infty e^{-ax} \sin sx \, dx = \frac{s}{a^2+s^2}$

(5) Find FCT and FST of $e^{-ax}$ and hence show that

(i) $\int_0^\infty \frac{\cos sx}{x^2+a^2} dx = \frac{\pi}{2a} e^{-as}$.

(ii) $\int_0^\infty \frac{x \sin sx}{x^2+a^2} dx = \frac{\pi}{2} e^{-as}$

Soln:

$f(x) = e^{-ax}$

(3) FCT of $e^{-ax}$:

$$F_c[f(x)] = F_c(s) = \frac{2}{\sqrt{2\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \frac{2}{\sqrt{2\pi}} \frac{a}{s^2+a^2}$$

FST of $e^{-ax}$:

$$F_s[f(x)] = F_s(s) = \frac{2}{\sqrt{2\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$= \frac{2}{\sqrt{2\pi}} \frac{s}{s^2+a^2}$$

(i) Using inverse Fourier cosine transforms,

$$f(x) = \frac{2}{\sqrt{2\pi}} \int_0^\infty F_c(s) \cos sx \, ds$$

$$e^{-ax} = \frac{2}{\sqrt{2\pi}} \int_0^\infty \frac{2}{\sqrt{2\pi}} \cdot \frac{a}{a^2+s^2} \cos sx \, ds$$

$$\frac{\pi}{2a} e^{-ax} = \int_0^\infty \frac{\cos sx}{s^2+a^2} \, ds$$

$$\therefore \boxed{\int_0^\infty \frac{\cos sx}{x^2+a^2} \, dx = \frac{\pi}{2a} e^{-ax}} \quad (x \leftrightarrow s).$$

(ii) Using inverse Fourier sine transforms,

$$f(x) = \frac{2}{\sqrt{2\pi}} \int_0^\infty F_s(s) \sin sx \, ds$$

$$e^{-ax} = \frac{2}{\sqrt{2\pi}} \int_0^\infty \frac{2}{\sqrt{2\pi}} \cdot \frac{s}{s^2+a^2} \sin sx \, ds$$

$$\int_0^\infty \frac{s \sin sx}{s^2+a^2} \, ds = \frac{\pi}{2} e^{-ax}$$

$$\boxed{\int_0^\infty \frac{x \sin sx}{x^2+a^2} \, dx = \frac{\pi}{2} e^{-as}} \quad (x \leftrightarrow s).$$

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(6) Find FCT of $e^{-2x} + 2e^{-4x}$

Soln:

(H) $$F_c[f(x)] = F_c(s) = \frac{2}{\sqrt{2\pi}} \left[ \frac{2}{s^2+4} + 2 \cdot \frac{4}{s^2+16} \right]$$

(7) Find FST of $\alpha e^{-ax} + \beta e^{-bx}$

Soln:

(6) $$F_S(s) = \frac{2}{\sqrt{2\pi}}\left[\alpha \cdot \frac{s}{s^2+a^2} + \beta \cdot \frac{s}{s^2+b^2}\right]$$

(8) Find FCT of $e^{-ax}\cos ax$.

Soln:

$$F_C(s) = \frac{2}{\sqrt{2\pi}}\int_0^\infty f(x)\cos sx\,dx$$

(7) $$= \frac{2}{\sqrt{2\pi}}\int_0^\infty e^{-ax}\cos ax\cos sx\,dx$$

$$= \frac{2}{\sqrt{2\pi}}\int_0^\infty e^{-ax}\cdot\frac{1}{2}\left[\cos(a+s)x + \cos(a-s)x\right]dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{a}{a^2+(a+s)^2} + \frac{a}{a^2+(a-s)^2}\right]$$

(9) Find FCT & FST of $\cosh x - \sinh x$.

Soln:

$$f(x) = \cosh x - \sinh x$$

(5) $$= \frac{e^x+e^{-x}}{2} - \frac{e^x-e^{-x}}{2}$$

$$f(x) = e^{-x}$$

$$F_c(s) = \frac{2}{\sqrt{2\pi}}\frac{1}{1+s^2}$$

$$F_s(s) = \frac{2}{\sqrt{2\pi}}\frac{s}{1+s^2}$$

(10) Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)^2}$

Numerator → $x$ → sine
↳ without $x$ → cosine
$\int$ → parsevals
without integral → inverse FT

Soln:

Take $f(x) = e^{-ax}$

(9) $F_c[f(x)] = F_c(s) = \frac{2}{\sqrt{2\pi}} \cdot \frac{a}{a^2+s^2}$

Using Parseval's identity,

$$\int_0^\infty [f(x)]^2\, dx = \int_0^\infty [F[f(x)]]^2\, ds$$

$$\int_0^\infty (e^{-ax})^2\, dx = \int_0^\infty \left[\frac{2}{\sqrt{2\pi}} \cdot \frac{a}{a^2+s^2}\right]^2 ds$$

$$\int_0^\infty e^{-2ax}\, dx = \int_0^\infty \frac{4}{2\pi} \cdot \frac{a^2}{(a^2+s^2)^2}\, ds$$

$$\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{\cancel{4}\, 2a^2}{\pi} \int_0^\infty \frac{ds}{(a^2+s^2)^2}$$

$$\frac{\pi}{4a^3} = \int_0^\infty \frac{ds}{(a^2+s^2)^2}$$

Put $s = x$, $ds = dx$

$$\boxed{\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}}$$

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(11) Evaluate $\int_0^\infty \frac{x^2\, dx}{(x^2+a^2)^2}$

(10) Soln:

$f(x) = e^{-ax}$

$F_s[f(x)] = \frac{2}{\sqrt{2\pi}} \cdot \frac{s}{s^2+a^2}$

Using Parseval's identity for Fourier Sine transforms,

$$\int_0^\infty [f(x)]^2 dx = \int_0^\infty [F_s(f(x))]^2 ds$$

$$\int_0^\infty (e^{-ax})^2 dx = \int_0^\infty \frac{4}{2\pi} \cdot \frac{s^2}{(s^2+a^2)^2} ds$$

$$\frac{1}{2a} \cdot \frac{\pi}{2} = \int_0^\infty \frac{s^2}{(s^2+a^2)^2} ds$$

Put $s = x \Rightarrow ds = dx$

$$\boxed{\int_0^\infty \frac{x^2}{(x^2+a^2)^2} dx = \frac{\pi}{4a}}$$

(12) Evaluate $\int_0^\infty \frac{x^2}{(x^2+4)(x^2+1)} dx$

Soln:

$$f(x) = e^{-ax} \quad ; \quad g(x) = e^{-bx}$$

$$= e^{-2x} \quad ; \quad = e^{-x}$$

$$F_s[f(x)] = \frac{2}{\sqrt{2\pi}} \frac{s}{s^2+4} \quad ; \quad F_s[g(x)] = \frac{2}{\sqrt{2\pi}} \cdot \frac{s}{s^2+1}$$

Using parseval's identity Property,

$$\int_0^\infty f(x) g(x) dx = \int_0^\infty F_s(s) \, G_s(s) ds$$

$$\int_0^\infty e^{-2x} e^{-x} dx = \int_0^\infty \frac{4}{2\pi} \cdot \frac{s}{s^2+4} \cdot \frac{s}{s^2+1} ds$$

$$\int_0^\infty e^{-3x} dx = \frac{2}{\pi} \int_0^\infty \frac{s^2}{(s^2+4)(s^2+1)} ds$$

$$\frac{1}{3}\cdot\frac{\pi}{2} = \int_0^\infty \frac{s^2}{(s^2+4)(s^2+1)}\,ds$$

Put $s = x \Rightarrow ds = dx$.

$$\therefore \boxed{\int_0^\infty \frac{x^2}{(x^2+4)(x^2+1)}\,dx = \frac{\pi}{6}}$$

(13) Evaluate $\displaystyle\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$

(11) Soln:

$$I = \frac{\pi}{2ab(a+b)}$$

(14) Evaluate $\displaystyle\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)}$

(12) Soln:

$$I = \frac{\pi}{2(a+b)}$$

(15) Find FST of $\dfrac{x}{x^2+a^2}$

Soln:

$$f(x) = e^{-ax}$$

$$F_s[f(x)] = F_s(s) = \frac{2}{\sqrt{2\pi}}\cdot\frac{s}{s^2+a^2}$$

(13)

Using inverse Fourier Sine transforms,

$$f(x) = \frac{2}{\sqrt{2\pi}}\int_0^\infty F_s(s)\sin sx\,ds$$

$$e^{-ax} = \frac{2}{\sqrt{2\pi}}\int_0^\infty \frac{2}{\sqrt{2\pi}}\frac{s}{s^2+a^2}\sin sx\,ds$$

$$e^{-ax} = \frac{4}{2\pi}\int_0^\infty \frac{s}{s^2+a^2}\sin sx\,ds$$