2. The frictional torque T of a disc diameter D rotating at a speed N in a fluid of viscosity μ and density ρ in a turbulent flow is given by $T = D^5 N^2 \rho \phi \left(\frac{\mu}{D^2 N \rho}\right)$ Prove it by Buckingham's π-theorem

(NOV 2003)

Buckingham's π-theorem.

Solution:

The Variables involved in analysis are

T, D, N, μ and ρ

The dimensions of each Variable are

Torque $T = ML^2T^{-2}$

Diameter $D = L$

Speed $N = T^{-1}$

Viscosity $\mu = ML^{-1}T^{-1}$

Density $\rho = ML^{-3}$

The functional relationship can be written as

$$T = f(D, N, \mu, \rho) \quad \text{——(1)}$$

$$f_1(T, D, N, \mu, \rho) = 0$$

The total number of Variables $n = 5$

Fundamental Variables $m = 3$

∴ The total number of π- terms $= n - m$

$= 5 - 3$

$= 2$

Each Variable has m+1 Variable

So, the functional equation interms of π-terms

$$f_1(\pi_1, \pi_2) = 0 \quad \text{------} (3)$$

$$\pi_1 = D^{a_1} \times N^{b_1} \times \rho^{c_1} \times T$$

$$\pi_2 = D^{a_2} \times N^{b_2} \times \rho^{c_2} \times \mu$$

π_1 - term

$$\pi_1 = D^{a_1} \times N^{b_1} \times \rho^{c_1} \times T$$

Now, the dimensionless equation becomes

$$M^0 L^0 T^0 = L^{a_1} \times (T^{-1})^{b_1} \times (ML^{-3})^{c_1} \times ML^2T^{-2}$$

Comparing exponents coefficient on both sides

for $M: 0 = c_1 + 1$ —— (i)

$L: 0 = a_1 - 3c_1 + 2$ —— (ii)

$T: 0 = -b_1 - 2$ —— (iii)

From (i) $c_1 = -1$

(ii) $a_1 = 3c_1 - 2 = 3(-1) - 2 = -5$

(iii) $b_1 = -2$

$$\pi_1 = D^{-5} \times N^{-2} \times \rho^{-1} \times T$$

$$\pi_1 = \frac{T}{D^5 N^2 \rho}.$$

π_2 - term

$$\pi_2 = D^{a_2} \times N^{b_2} \times \rho^{c_2} \times \mu$$

Now, the dimension less equation becomes

$$M^0 L^0 T^0 = L^{a_2} \times (T^{-1})^{b_2} \times (ML^{-3})^{c_2} \times ML^{-1}T^{-1}$$

Comparing exponents coefficient on both sides

$$M \quad 0 = c_2 + 1 \quad \text{—— iv}$$

$$0 = a_2 - 3c_2 - 1 \quad \text{—— v}$$

$$0 = -b_2 - 1 \quad \text{—— vi}$$

From (iv) $c_2 = -1$

(v) $a_2 = 3c_2 + 1 = 3(-1) + 1 = -2$

(vi) $b_2 = -1$

$$\pi_2 = D^{-2} N^{-1} \rho^{-1} \mu = \frac{\mu}{D^2 N \rho}$$

Substituting values of π_1 and π_2 in (3)

$$f\left(\frac{T}{D^5 N^5 \rho} \cdot \frac{\mu}{D^2 N \rho}\right) = 0$$

$$T = D^5 N^2 \rho \, \phi\left(\frac{\mu}{D^2 N \rho}\right)$$

Hence it is prooved.