Problems based on the interval $(-l, l)$

Even function:

A function is said to be even if $f(x) = f(-x)$. i.e., If $f(x)$ is an even function,

$$\int_{-l}^{l} f(x)\,dx = 2\int_{0}^{l} f(x)\,dx$$

Odd function:

A function is said to be odd if $f(x) = -f(-x)$ or $f(-x) = -f(x)$. If $f(x)$ is an odd function,

$$\int_{-l}^{l} f(x)\,dx = 0.$$

Note:

If $f(x)$ does not satisfies even and odd function then it is called neither even nor odd function.

Example:

1. $f(x) = x^2$ in $(-\pi, \pi)$ — Even
2. $f(x) = x\cos x$ in $(-l, l)$ — Odd
3. $f(x) = x\sin x$ in $(-\pi, \pi)$ — Even
4. $f(x) = |x|$ in $(-2, 2)$ — Even
5. $f(x) = (l-x)^2$ in $(-l, l)$ — Neither even nor odd
6. $f(x) = x + x^2$ in $(-\pi, \pi)$ — Neither even nor odd
7. $f(x) = x - x^2$ in $(-2, 2)$ — Neither even nor odd

Note:

For split up problem,

$$f(x) = \begin{cases} \varphi_1(x) \\ \varphi_2(x) \end{cases}$$

Even $\rightarrow$ $\varphi_1(x) = \varphi_2(-x)$

Odd $\rightarrow$ $\varphi_1(x) = -\varphi_2(-x)$.

Example:

* $$f(x) = \begin{cases} l + x, & -l < x < 0 \\ l - x, & 0 < x < l \end{cases}$$

$\varphi_1(x) = l + x$, $\varphi_1(-x) = l - x = \varphi_2(x)$

$\varphi_2(x) = l - x$, $\varphi_2(-x) = l + x = \varphi_1(x)$.

$\therefore f(x)$ is an even function.

* $$f(x) = \begin{cases} -k, & -\pi \le x \le 0 \\ k, & 0 \le x \le \pi \end{cases}$$

$\varphi_1(x) = -k$, $\varphi_1(-x) = -k = -\varphi_2(x)$.

$\varphi_2(x) = k$, $\varphi_2(-x) = k = -\varphi_1(x)$

$\therefore f(x)$ is an odd function.

Note:

* Even $\times$ Even $=$ Even
* Odd $\times$ Odd $=$ Even
* Even $\times$ Odd $=$ Odd
* Odd $\times$ Even $=$ Odd.

Interval $(-l, l)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$f(x)$ is an even function	$f(x)$ is an odd function	$f(x)$ is neither even nor odd
$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$ $= \frac{2}{l}\int_{0}^{l} f(x)\,dx$	$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$ $\boxed{a_0 = 0}$	$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$
$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$ $= \frac{2}{l}\int_{0}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$	$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$ $\boxed{a_n = 0}$	$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$
$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$ $\boxed{b_n = 0}$	$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$ $b_n = \frac{2}{l}\int_{0}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$	$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

1. Find the Fourier series for the function

$$f(x) = \begin{cases} L+x, & (-L, 0) \\ L-x, & (0, L) \end{cases}$$

Soln:

Here $l = L$

$\varphi_1(x) = L + x$

$\varphi_1(-x) = L - x = \varphi_2(x)$

$\varphi_2(x) = L - x$

$\varphi_2(-x) = L-(-x) = L + x = \varphi_1(x)$

$\therefore f(x)$ is an even function.

$$\boxed{b_n = 0}$$

Fourier Series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \longrightarrow (1)$$

To find a_0:

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$= \frac{2}{L}\int_{0}^{L} f(x)\,dx$$

$$= \frac{2}{L}\int_{0}^{L} (L-x)\,dx$$

$$= \frac{2}{L}\left[Lx - \frac{x^2}{2}\right]_0^L$$

$$\boxed{a_0 = L}$$

To find a_n :

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{L}\int_{0}^{L}(L-x)\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\left[\frac{L}{n\pi}(L-x)\sin\left(\frac{n\pi x}{L}\right) - \frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right)\right]_0^L$$

$$= \frac{2}{L}\left[\frac{-L^2}{n^2\pi^2}\cos n\pi + \frac{L^2}{n^2\pi^2}\cos 0\right]$$

$$= \frac{2}{L}\cdot\frac{L^2}{n^2\pi^2}\left[1-(-1)^n\right]$$

$$\boxed{a_n = \frac{2L}{n^2\pi^2}\left[1-(-1)^n\right]}$$

Subs a_0, a_n and b_n in (1),

$$f(x) = \frac{L}{2} + \sum_{n=1}^{\infty}\frac{2L}{n^2\pi^2}\left[1-(-1)^n\right]\cos\left(\frac{n\pi x}{L}\right)$$

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(2) $f(x) = |x|$ in $-l \le x \le l$.

Soln:

$f(x)$ is an even function

$b_n = 0$.

$a_0 = l$

$$a_n = \frac{2l}{n^2\pi^2}\left[(-1)^n - 1\right]$$

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(3) $f(x) = |x|$ in $-\pi \le x \le \pi$

Soln:

$f(x)$ is an even function

$$b_n = 0$$

$$a_0 = \pi$$

$$a_n = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

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4) $f(x) = \begin{cases} 1+x, & -2 \le x \le 0 \\ 1-x, & 0 \le x \le 2 \end{cases}$

Soln:

$f(x)$ is an even fn.

$$b_n = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} \rightarrow (1)$$

To find a_0:

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$= \frac{2}{l}\int_{0}^{l} f(x)\,dx$$

$$= \frac{2}{2}\int_{0}^{2} (1-x)\,dx$$

$$\boxed{a_0 = 0}.$$

To find a_n:

$$a_n = \int_{0}^{2} (1-x)\cos\left(\frac{n\pi x}{2}\right) dx$$

$$\boxed{a_n = \frac{4}{\pi^2 n^2}\left[1-(-1)^n\right]}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{4}{\pi n^2}\left[1-(-1)^n\right]\cos\left(\frac{n\pi x}{2}\right)$$

(5) $f(x) = x - x^2, \ -l \le x \le l.$

Soln:

$f(x)$ is neither even nor odd function.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \rightarrow (1)$$

To find a_0:

$$a_0 = \frac{1}{l}\int_{-l}^{l} (x - x^2)\,dx$$

$$\boxed{a_0 = -\frac{2l^2}{3}}$$

To find a_n:

$$a_n = \frac{1}{l}\int_{-l}^{l} (x - x^2)\cos\frac{n\pi x}{l}\,dx$$

$$\boxed{a_n = \frac{-4l^2(-1)^n}{n^2\pi^2}}$$

To find b_n:

$$b_n = \frac{1}{l}\int_{-l}^{l} (x - x^2)\sin\frac{n\pi x}{l}\,dx$$

$$\boxed{b_n = \frac{-2l(-1)^n}{n\pi}}$$

Subs a_0, a_n & b_n in (1),

$$f(x) = -\frac{l^2}{3} + \sum_{n=1}^{\infty} \frac{-4l^2(-1)^n}{n^2\pi^2}\cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} \frac{-2l(-1)^n}{n\pi}\sin\left(\frac{n\pi x}{l}\right).$$