# PROPERTIES :

## Change of Scale property :

If $L\{f(t)\} = F(S)$, then

$$L[f(at)] = \frac{1}{a} F(S/a).$$

**Proof :**

We know that,

$$L[f(at)] = \int_0^\infty e^{-st} f(at)\, dt$$

put $at = x \Rightarrow t = x/a$

$dt = dx/a$

$$L[f(at)] = \int_0^\infty e^{-s(x/a)} f(x) \frac{dx}{a}$$

$$= \frac{1}{a} \int_0^\infty e^{-s(x/a)} f(x)\, dx$$

$$= \frac{1}{a} \int_0^\infty e^{-(S/a)x} f(x)\, dx$$

$$= \frac{1}{a} \int_0^\infty e^{-(S/a)t} f(t)\, dt$$

$$= \frac{1}{a} F(S/a)$$

## First Shifting property:

If $L\{f(t)\} = F(s)$ then

(i) $L[e^{-at} f(t)] = \{L[f(t)]\}_{s \to s+a} = F(s+a)$

(ii) $L[e^{at} f(t)] = \{L[f(t)]\}_{s \to s-a} = F(s-a)$

**Proof:**

(i) We know that,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

$$L[e^{-at} f(t)] = \int_0^\infty e^{-st} [e^{-at} f(t)]\,dt$$

$$= \int_0^\infty e^{-(s+a)t} f(t)\,dt$$

$$= F(s+a)$$

(ii) $$L[e^{at} f(t)] = \int_0^\infty e^{-st} [e^{at} f(t)]\,dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

$$= F(s-a).$$

## Second Shifting property:

If $L\{f(t)\} = F(s)$ and $g(t) = \begin{cases} f(t-a), & t>a \\ 0, & t<a \end{cases}$

then $L[g(t)] = e^{-as} F(s)$.

**Proof:**

$$L[g(t)] = \int_0^\infty e^{-st} g(t)\,dt$$

$$= \int_0^a e^{-st} g(t)\,dt + \int_a^\infty e^{-st} g(t)\,dt$$

$$L[g(t)] = 0 + \int_a^\infty e^{-st} f(t-a)\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

Put $t - a = u \Rightarrow dt = du$

When $t = a \Rightarrow u = 0$

$t \to \infty \Rightarrow u \to \infty$

$$L[g(t)] = \int_0^\infty e^{-s(u+a)} f(u)\,du$$

$$= \int_0^\infty e^{-us} e^{-as} f(u)\,du$$

$$= e^{-as} \int_0^\infty e^{-us} f(u)\,du$$

$$= e^{-as} \int_0^\infty e^{-st} f(t)\,dt \quad \text{Replace } u \to t$$

$$L[g(t)] = e^{-as} F(s).$$

**Laplace transforms of derivatives :**

If $L[f(t)] = F(s)$ then

$$L[f'(t)] = s\,F(s) - f(0).$$

**Proof :**

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\,dt$$

Integrating by parts we get,

$$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty f(t)(-se^{-st})\,dt$$

$$= [e^{-\infty} f(\infty) - e^{0} f(0)] + s \int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + s\, L\{f(t)\}$$

$$= s\,F(s) - f(0).$$

**Corollary:**

Let $f''(t) = s^2 F(s) - s f(0) - f'(0)$

Let $L[g'(t)] = s\,G(s) - g(0)$.

We know that,

$$L[f'(t)] = s\,L[f(t)] - f(0).$$

Replace $f(t) \to f'(t)$ & $f'(t) \to f''(t)$ & $f(0) \to f'(0)$.

$$\Rightarrow L[f''(t)] = s\,L[f'(t)] - f'(0)$$

$$= s\,[s(L[f(t)] - f(0))] - f'(0)$$

$$= s^2 L[f(t)] - s f(0) - f'(0)$$

$$= s^2 F(s) - s f(0) - f'(0).$$

**Laplace Transform of integrals:**

If $L[f(t)] = F(s)$ then $L\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$

**Proof:**

Let $g(t) = \int_0^t f(t)\,dt$ and $g(0) = 0$

then $g'(t) = f(t)$

WKT $L[g'(t)] = s\,L(g(t)) - g(0)$

$$= s\,L(g(t))$$

$$\Rightarrow L[g(t)] = \frac{1}{s} L[g'(t)]$$

$$\Rightarrow L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s} L[f(t)] \quad \left\{ \because g(t) = \int_0^t f(t) \right.$$

$$g'(t) = f(t)$$

$$\Rightarrow L\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$$

## Derivative of Laplace Transform (or) Laplace transform of $t\,f(t)$:

If $L[f(t)] = F(s)$ then

$$L[t\,f(t)] = -\frac{d}{ds} F(s)$$

**Proof:**

We know that,

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\frac{d}{ds} F(s) = \frac{d}{ds} \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty \frac{\partial}{\partial s}(e^{-st}) f(t)\,dt$$

$$= \int_0^\infty -t e^{-st} f(t)\,dt$$

$$= -\int_0^\infty e^{-st}\, t\, f(t)\,dt$$

$$= -L[t\,f(t)]$$

$$\Rightarrow L[t\,f(t)] = -\frac{d}{ds}[F(s)]$$

In general,

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)]$$

## Problems :

### Change of Scale property :

① Find $L[\sinh 3t]$ by using change of scale property

Soln:

$$L[\sinh t] = \frac{1}{s^2-1} = F(s)$$

$$L[\sinh 3t] = \frac{1}{3} F\left(\frac{s}{3}\right)$$

$$= \frac{1}{3} \frac{1}{\left(\frac{s}{3}\right)^2 - 1}$$

$$= \frac{1}{3}\left(\frac{9}{s^2-9}\right)$$

$$= \frac{3}{s^2-9}$$

② Find $L(\cos 5t)$ using change of scale property ?

Soln:

$$L(\cos t) = \frac{s}{s^2+1} = F(s)$$

$$L(\cos 5t) = \frac{1}{5} F\left(\frac{s}{5}\right)$$

$$= \frac{1}{5}\left[\frac{s/5}{\left(\frac{s}{5}\right)^2+1}\right]$$

$$= \frac{1}{5}\left[\frac{5s}{s^2+25}\right]$$

$$= \frac{s}{s^2+25}$$

(3) Given $L[f(t)] = \dfrac{s^2-s+1}{(2s+1)^2(s-1)}$ applying the Change of scale property show that

$$L[f(2t)] = \frac{s^2-2s+4}{4(s+1)^2(s-2)}$$

Soln:

$$L[f(t)] = \frac{s^2-s+1}{(2s+1)^2(s-1)} = F(s)$$

$$L[f(2t)] = \frac{1}{2} F(s/2)$$

$$= \frac{1}{2}\left[\frac{(s/2)^2-(s/2)+1}{(2s/2+1)^2\left(\frac{s}{2}-1\right)}\right]$$

$$= \frac{1}{2}\left[\frac{\frac{s^2-2s+4}{4}}{(s+1)^2(s-2)/2}\right]$$

$$= \frac{1}{4}\cdot\frac{s^2-2s+4}{(s+1)^2(s-2)}$$

(4) Find $L[e^{5t}]$ applying change of scale property.

Soln:

$$L(e^t) = \frac{1}{s-1} = F(s)$$

$$L(e^{5t}) = \frac{1}{5} F(s/5)$$

$$= \frac{1}{5}\cdot\frac{1}{s/5-1}$$

$$= \frac{1}{5}\cdot\frac{5}{s-5}$$

$$= \frac{1}{s-5}$$

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## First Shifting theorem:

1) Find $L[e^{-3t}\sin^2 t]$

Proof:

$$L[e^{-at}f(t)] = F(s+a)$$

$$L[e^{-3t}\sin^2 t] = L[\sin^2 t]_{s\to s+3}$$

$$= L\left[\frac{1-\cos 2t}{2}\right]_{s\to s+3}$$

$$= \frac{1}{2}\{L(1) - L(\cos 2t)\}_{s\to s+3}$$

$$= \frac{1}{2}\left\{\frac{1}{s} - \frac{s}{s^2+4}\right\}_{s\to s+3}$$

$$= \frac{1}{2}\left\{\frac{1}{s+3} - \frac{s+3}{(s+3)^2+4}\right\}$$

$$= \frac{1}{2}\left\{\frac{4}{(s+3)[(s+3)^2+4]}\right\}$$

$$= \frac{2}{(s+3)[(s+3)^2+4]}$$

2) Find $L(t^2 e^{-2t})$.

Soln:

$$L[e^{-at}f(t)] = F(s+a)$$

$$L[e^{-2t}t^2] = [L(t^2)]_{s\to s+2}$$

$$= \left[\frac{2}{s^3}\right]_{s\to s+2}$$

$$= \frac{2}{(s+2)^3}$$

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(3) Find $L[e^{2t} \cos 5t]$.

Soln:

$$L[e^{2t}\cos 5t] = L[\cos 5t]_{s \to s-2}$$

$$= \left[\frac{s}{s^2+25}\right]_{s \to s-2}$$

$$= \frac{s-2}{(s-2)^2+25}$$

## Second Shifting theorem:

(1) Find $L[f(t)]$ where $f(t) = \begin{cases} 0, & 0 < t < 2 \\ 3, & t > 2 \end{cases}$

Soln:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^2 e^{-st} f(t)\,dt + \int_2^\infty e^{-st} f(t)\,dt$$

$$= 0 + \int_2^\infty e^{-st} \cdot 3\,dt$$

$$= 3\left[\frac{e^{-st}}{-s}\right]_2^\infty$$

$$= \frac{-3}{s}\left[e^{-\infty} - e^{-2s}\right]$$

$$= \frac{3e^{-2s}}{s}$$

(2) Find the Laplace transform of

$$f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$$

Soln:

$$\left[ f(t) = \begin{matrix} \sin t, & 0 < t < \pi \\ = 0, & \pi < t < 2\pi \end{matrix} \right\}$$

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$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\pi e^{-st} f(t)\,dt + \int_\pi^\infty e^{-st}\cdot 0\,dt$$

$$= \left[\frac{e^{-st}}{s^2+1}(-s\sin t - \cos t)\right]_0^\pi \quad \left\{\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right\}$$

$$= \frac{e^{-s\pi}}{s^2+1}(-s\sin\pi - \cos\pi) + \frac{1}{s^2+1}$$

$$= \frac{e^{-\pi s}}{s^2+1} + \frac{1}{s^2+1} = \frac{1+e^{-\pi s}}{s^2+1}$$

## Laplace Transforms of Derivatives :

(1) Find $L[t\sin at]$

Soln:

$L[t\sin at] =$

$f(t) = t\sin at$

$f'(t) = at\cos at + \sin at$

$f''(t) = a[-at\sin at + \cos at] + a\cos at$

$= 2a\cos at - a^2 t\sin at$

$f(0) = 0, \quad f'(0) = 0$

$L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0)$

$L[2a\cos at - a^2 t\sin at] = s^2 L[t\sin at] - s(0) - 0$

$\Rightarrow 2aL(\cos at) - a^2 L(t\sin at) = s^2 L(t\sin at)$

$\Rightarrow (s^2+a^2) L(t\sin at) = 2aL(\cos at)$

$\Rightarrow (s^2+a^2) L(t\sin at) = 2a\cdot\frac{s}{a^2+s^2}$

$$L(t\sin at) = \frac{2as}{(s^2+a^2)^2}$$

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① Find $L[t\cos at]$

Soln:

$$\because L[t f(t)] = -\frac{d}{ds}[L(f(t))]$$

$$L[t\cos at] = -\frac{d}{ds}[L(\cos at)]$$

$$= -\frac{d}{ds}\left[\frac{s}{s^2+a^2}\right]$$

$$= -\left\{\frac{s^2+a^2-s(2s)}{(s^2+a^2)^2}\right\}$$

$$= -\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{s^2-a^2}{(s^2+a^2)^2}$$

② Find $L[te^{2t}\sin 3t]$

Soln:

$$L[te^{2t}\sin 3t] = -\frac{d}{ds}\{L[e^{2t}\sin 3t]\}$$

$$= -\frac{d}{ds}\{L(\sin 3t)\}_{s\to s-2}$$

$$= -\frac{d}{ds}\left\{\left(\frac{3}{s^2+9}\right)_{s\to s-2}\right\}$$

$$= -\left\{\frac{0-3(2s)}{(s^2+9)^2}\right\}_{s\to s-2}$$

$$= \left\{\frac{6s}{(s^2+9)^2}\right\}_{s\to s-2}$$

$$= \frac{6(s-2)}{[(s-2)^2+9]^2}$$

$$= \frac{6(s-2)}{(s^2-4s+13)^2}$$

③ Find $L[t^2 e^{-2t} \cos t]$

Soln:

$$L[t^2 e^{-2t}\cos t] = (-1)^2 \frac{d^2}{ds^2}\{L(e^{-2t}\cos t)\}$$

$$= \frac{d^2}{ds^2}\{L(\cos t)_{s\to s+2}\}$$

$$= \frac{d^2}{ds^2}\left\{\frac{s}{s^2+1}\right\}_{s\to s+2}$$

$$= \frac{d}{ds}\left\{\frac{s^2+1-s(2s)}{(s^2+1)^2}\right\}_{s\to s+2}$$

$$= \frac{d}{ds}\left\{\frac{1-s^2}{(s^2+1)^2}\right\}_{s\to s+2}$$

$$= \left\{\frac{(s^2+1)^2(-2s)-(1-s^2)2(s^2+1)(2s)}{(s^2+1)^4}\right\}_{s\to s+2}$$

$$= \left\{\frac{(s^2+1)(-2s)-4s(1-s^2)}{(s^2+1)^3}\right\}_{s\to s+2}$$

$$= \frac{[(s+2)^2+1][-2(s+2)]-4(s+2)[1-(s+2)^2]}{[(s+2)^2+1]^3}$$

$$= \frac{(s^2+4s+5)(-2s-4)+(4s+8)(s^2+4s+3)}{[(s+2)^2+1]^3}$$

$$= \frac{2s^3+12s^2+18s+4}{(s^2+4s+5)^3}$$

(4) Find $L\left[\frac{\sin 3t}{t}\right]$

Soln:

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds = \int_s^\infty L[f(t)]\,ds$$

$$L\left[\frac{\sin 3t}{t}\right] = \int_s^\infty L(\sin 3t)\,ds$$

$$= \int_s^\infty \left(\frac{3}{s^2+9}\right) ds$$

$$= \int_s^\infty \frac{3}{s^2+3^2}\,ds$$

$$= 3 \cdot \frac{1}{3}\left[\tan^{-1}\left(\frac{s}{3}\right)\right]_s^\infty \qquad \left(\because \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right)$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s/3)$$

$$= \pi/2 - \tan^{-1}(s/3)$$

$$= \cot^{-1}(s/3).$$