

Propositional and First-Order Logic

Chapter 7.4–7.8, 8.1–8.3, 8.5

Logic roadmap overview

- Propositional logic (review)
- Problems with propositional logic
- First-order logic (review)
 - Properties, relations, functions, quantifiers, ...
 - Terms, sentences, wffs, axioms, theories, proofs, ...
- Extensions to first-order logic
- Logical agents
 - Reflex agents
 - Representing change: situation calculus, frame problem
 - Preferences on actions
 - Goal-based agents

Disclaimer

“Logic, like whiskey, loses its beneficial effect when taken in too large quantities.”

- *Lord Dunsany*

Propositional Logic: Review

Big Ideas

- Logic is a great knowledge representation language for many AI problems
- **Propositional logic** is the simple foundation and fine for some AI problems
- **First order logic** (FOL) is much more expressive as a KR language and more commonly used in AI
- There are many variations: horn logic, higher order logic, three-valued logic, probabilistic logics, etc.

Propositional logic

- **Logical constants:** true, false
- **Propositional symbols:** P, Q,... (**atomic sentences**)
- **Wrapping parentheses:** (...)
- Sentences are combined by **connectives**:
 - \wedge and [conjunction]
 - \vee or [disjunction]
 - \Rightarrow implies [implication / conditional]
 - \Leftrightarrow is equivalent [biconditional]
 - \neg not [negation]
- **Literal:** atomic sentence or negated atomic sentence
P, $\neg P$

Examples of PL sentences

- $(P \wedge Q) \rightarrow R$
“If it is hot and humid, then it is raining”
- $Q \rightarrow P$
“If it is humid, then it is hot”
- Q
“It is humid.”
- We’re free to choose better symbols, btw:
Ho = “It is hot”
Hu = “It is humid”
R = “It is raining”

Propositional logic (PL)

- Simple language for showing key ideas and definitions
- User defines set of propositional symbols, like P and Q
- User defines **semantics** of each propositional symbol:
 - P means “It is hot”, Q means “It is humid”, etc.
- A sentence (well formed formula) is defined as follows:
 - A symbol is a sentence
 - If S is a sentence, then $\neg S$ is a sentence
 - If S is a sentence, then (S) is a sentence
 - If S and T are sentences, then $(S \vee T)$, $(S \wedge T)$, $(S \rightarrow T)$, and $(S \leftrightarrow T)$ are sentences
 - A sentence results from a finite number of applications of the rules

Some terms

- The meaning or **semantics** of a sentence determines its **interpretation**
- Given the truth values of all symbols in a sentence, it can be “evaluated” to determine its **truth value** (True or False)
- A **model** for a KB is a *possible world* – an assignment of truth values to propositional symbols that makes each sentence in the KB True

Model for a KB

- Let the KB be $[P \wedge Q \rightarrow R, Q \rightarrow P]$
- What are the possible models? Consider all possible assignments of T|F to P, Q and R and check truth tables
 - **FFF: OK**
 - **FFT: OK**
 - FTF: NO
 - FTT: NO
 - **TFF: OK**
 - **TFT: OK**
 - TTF: NO
 - **TTT: OK**
- If KB is $[P \wedge Q \rightarrow R, Q \rightarrow P, Q]$, then the only model is TTT

P: it's hot

Q: it's humid

R: it's raining

More terms

- A **valid sentence** or **tautology** is a sentence that is True under all interpretations, no matter what the world is actually like or what the semantics is. Example: “It’s raining or it’s not raining”
- An **inconsistent sentence** or **contradiction** is a sentence that is False under all interpretations. The world is never like what it describes, as in “It’s raining and it’s not raining.”
- **P entails Q**, written $P \models Q$, means that whenever P is True, so is Q. In other words, all models of P are also models of Q.

Truth tables

- Truth tables are used to define logical connectives
- and to determine when a complex sentence is true given the values of the symbols in it

Truth tables for the five logical connectives

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
<i>False</i>	<i>False</i>	<i>True</i>	<i>False</i>	<i>False</i>	<i>True</i>	<i>True</i>
<i>False</i>	<i>True</i>	<i>True</i>	<i>False</i>	<i>True</i>	<i>True</i>	<i>False</i>
<i>True</i>	<i>False</i>	<i>False</i>	<i>False</i>	<i>True</i>	<i>False</i>	<i>False</i>
<i>True</i>	<i>True</i>	<i>False</i>	<i>True</i>	<i>True</i>	<i>True</i>	<i>True</i>

Example of a truth table used for a complex sentence

P	H	$P \vee H$	$(P \vee H) \wedge \neg H$	$((P \vee H) \wedge \neg H) \Rightarrow P$
<i>False</i>	<i>False</i>	<i>False</i>	<i>False</i>	<i>True</i>
<i>False</i>	<i>True</i>	<i>True</i>	<i>False</i>	<i>True</i>
<i>True</i>	<i>False</i>	<i>True</i>	<i>True</i>	<i>True</i>
<i>True</i>	<i>True</i>	<i>True</i>	<i>False</i>	<i>True</i>

On the implies connective: $P \rightarrow Q$

- Note that \rightarrow is a logical connective
- So $P \rightarrow Q$ is a logical sentence and has a truth value, i.e., is either true or false
- If we add this sentence to the KB, it can be used by an inference rule, *Modes Ponens*, to derive/infer/prove Q if P is also in the KB
- Given a KB where $P = \text{True}$ and $Q = \text{True}$, we can also derive/infer/prove that $P \rightarrow Q$ is True

$P \rightarrow Q$

- When is $P \rightarrow Q$ true? Check all that apply
 - $P=Q=\text{true}$
 - $P=Q=\text{false}$
 - $P=\text{true}, Q=\text{false}$
 - $P=\text{false}, Q=\text{true}$

$P \rightarrow Q$

- When is $P \rightarrow Q$ true? Check all that apply
 - $P=Q=\text{true}$
 - $P=Q=\text{false}$
 - $P=\text{true}, Q=\text{false}$
 - $P=\text{false}, Q=\text{true}$
- We can get this from the truth table for \rightarrow
- Note: in FOL it's much harder to prove that a conditional true.
 - Consider proving $\text{prime}(x) \rightarrow \text{odd}(x)$

Inference rules

- **Logical inference** creates new sentences that logically follow from a set of sentences (KB)
- An inference rule is **sound** if every sentence X it produces when operating on a KB logically follows from the KB
 - i.e., inference rule creates no contradictions
- An inference rule is **complete** if it can produce every expression that logically follows from (is entailed by) the KB.
 - Note analogy to complete search algorithms

Sound rules of inference

- Here are some examples of sound rules of inference
- Each can be shown to be sound using a truth table

<u>RULE</u>	<u>PREMISE</u>	<u>CONCLUSION</u>
Modus Ponens	$A, A \rightarrow B$	B
And Introduction	A, B	$A \wedge B$
And Elimination	$A \wedge B$	A
Double Negation	$\neg\neg A$	A
Unit Resolution	$A \vee B, \neg B$	A
Resolution	$A \vee B, \neg B \vee C$	$A \vee C$

Soundness of modus ponens

A	B	$A \rightarrow B$	OK?
True	True	True	✓
True	False	False	✓
False	True	True	✓
False	False	True	✓

Resolution

- **Resolution** is a valid inference rule producing a new clause implied by two clauses containing *complementary literals*
 - A literal is an atomic symbol or its negation, i.e., P , $\sim P$
- Amazingly, this is the only inference rule you need to build a sound and complete theorem prover
 - Based on proof by contradiction and usually called resolution refutation
- The resolution rule was discovered by Alan Robinson (CS, U. of Syracuse) in the mid 60s

Resolution

- A KB is actually a set of sentences all of which are true, i.e., a conjunction of sentences.
- To use resolution, put KB into *conjunctive normal form* (CNF), where each sentence written as a disjunction of (one or more) literals

Tautologies

$$(A \rightarrow B) \leftrightarrow (\sim A \vee B)$$

$$(A \vee (B \wedge C)) \leftrightarrow (A \vee B) \wedge (A \vee C)$$

Example

- KB: $[P \rightarrow Q, Q \rightarrow R \wedge S]$
- KB in CNF: $[\sim P \vee Q, \sim Q \vee R, \sim Q \vee S]$
- Resolve KB(1) and KB(2) producing: $\sim P \vee R$ (*i.e.*, $P \rightarrow R$)
- Resolve KB(1) and KB(3) producing: $\sim P \vee S$ (*i.e.*, $P \rightarrow S$)
- New KB: $[\sim P \vee Q, \sim Q \vee \sim R \vee \sim S, \sim P \vee R, \sim P \vee S]$

Soundness of the resolution inference rule

α	β	γ	$\alpha \vee \beta$	$\neg\beta \vee \gamma$	$\alpha \vee \gamma$
<i>False</i>	<i>False</i>	<i>False</i>	<i>False</i>	<i>True</i>	<i>False</i>
<i>False</i>	<i>False</i>	<i>True</i>	<i>False</i>	<i>True</i>	<i>True</i>
<i>False</i>	<i>True</i>	<i>False</i>	<i>True</i>	<i>False</i>	<i>False</i>
<u><i>False</i></u>	<u><i>True</i></u>	<u><i>True</i></u>	<u><i>True</i></u>	<u><i>True</i></u>	<u><i>True</i></u>
<u><i>True</i></u>	<u><i>False</i></u>	<u><i>False</i></u>	<u><i>True</i></u>	<u><i>True</i></u>	<u><i>True</i></u>
<u><i>True</i></u>	<u><i>False</i></u>	<u><i>True</i></u>	<u><i>True</i></u>	<u><i>True</i></u>	<u><i>True</i></u>
<i>True</i>	<i>True</i>	<i>False</i>	<i>True</i>	<i>False</i>	<i>True</i>
<u><i>True</i></u>	<u><i>True</i></u>	<u><i>True</i></u>	<u><i>True</i></u>	<u><i>True</i></u>	<u><i>True</i></u>

From the rightmost three columns of this truth table, we can see that

$$(\alpha \vee \beta) \wedge (\beta \vee \gamma) \leftrightarrow (\alpha \vee \gamma)$$

is valid (i.e., always true regardless of the truth values assigned to α , β and γ)

Proving things

- A **proof** is a sequence of sentences, where each is a premise or is derived from earlier sentences in the proof by an inference rule
- The last sentence is the **theorem** (also called goal or query) that we want to prove
- Example for the “weather problem”

1	Hu	premise	“It’s humid”
2	$Hu \rightarrow Ho$	premise	“If it’s humid, it’s hot”
3	Ho	modus ponens(1,2)	“It’s hot”
4	$(Ho \wedge Hu) \rightarrow R$	premise	“If it’s hot & humid, it’s raining”
5	$Ho \wedge Hu$	and introduction(1,3)	“It’s hot and humid”
6	R	modus ponens(4,5)	“It’s raining”

Horn sentences

- A **Horn sentence** or **Horn clause** has the form:

$$P_1 \wedge P_2 \wedge P_3 \dots \wedge P_n \rightarrow Q_m \text{ where } n \geq 0, m \in \{0, 1\}$$

- Note: a conjunction of 0 or more symbols to left of \rightarrow and 0-1 symbols to right
- Special cases:
 - $n=0, m=1$: P (assert P is true)
 - $n>0, m=0$: $P \wedge Q \rightarrow$ (constraint: both P and Q can't be true)
 - $n=0, m=0$: (well, there is nothing there!)
- Put in CNF: each sentence is a disjunction of literals with at most one non-negative literal

$$\neg P_1 \vee \neg P_2 \vee \neg P_3 \dots \vee \neg P_n \vee Q$$

$$(P \rightarrow Q) = (\neg P \vee Q)$$

Significance of Horn logic

- We can also have horn sentences in FOL
- Reasoning with horn clauses is much simpler
 - Satisfiability of a propositional KB (i.e., finding values for a symbols that will make it true) is NP complete
 - Restricting KB to horn sentences, satisfiability is in P
- For this reason, FOL Horn sentences are the basis for Prolog and Datalog
- What Horn sentences give up are handling, in a general way, (1) negation and (2) disjunctions

Entailment and derivation

- **Entailment: $KB \models Q$**

- Q is entailed by KB (set sentences) iff there is no logically possible world where Q is false while all the sentences in KB are true
- Or, stated positively, Q is entailed by KB iff the conclusion is true in every logically possible world in which all the premises in KB are true

- **Derivation: $KB \vdash Q$**

- We can derive Q from KB if there's a proof consisting of a sequence of valid inference steps starting from the premises in KB and resulting in Q

Two important properties for inference

Soundness: If $KB \vdash Q$ then $KB \models Q$

- If Q is derived from KB using a given set of rules of inference, then Q is entailed by KB
- Hence, inference produces only real entailments, or any sentence that follows deductively from the premises is valid

Completeness: If $KB \models Q$ then $KB \vdash Q$

- If Q is entailed by KB , then Q can be derived from KB using the rules of inference
- Hence, inference produces all entailments, or all valid sentences can be proved from the premises

Problems with Propositional Logic

Propositional logic: pro and con

- Advantages
 - Simple KR language sufficient for some problems
 - Lays the foundation for higher logics (e.g., FOL)
 - Reasoning is decidable, though NP complete, and efficient techniques exist for many problems
- Disadvantages
 - Not expressive enough for most problems
 - Even when it is, it can very “un-concise”

PL is a weak KR language

- Hard to identify “individuals” (e.g., Mary, 3)
- Can’t directly talk about properties of individuals or relations between individuals (e.g., “Bill is tall”)
- Generalizations, patterns, regularities can’t easily be represented (e.g., “all triangles have 3 sides”)
- First-Order Logic (FOL) is expressive enough to represent this kind of information using relations, variables and quantifiers, e.g.,
 - *Every elephant is gray*: $\forall x (\text{elephant}(x) \rightarrow \text{gray}(x))$
 - *There is a white alligator*: $\exists x (\text{alligator}(X) \wedge \text{white}(X))$

PL Example

- Consider the problem of representing the following information:
 - Every person is mortal.
 - Confucius is a person.
 - Confucius is mortal.
- How can these sentences be represented so that we can infer the third sentence from the first two?

PL Example

- In PL we have to create propositional symbols to stand for all or part of each sentence, e.g.:
P = “person”; Q = “mortal”; R = “Confucius”
- The above 3 sentences are represented as:
 $P \rightarrow Q; R \rightarrow P; R \rightarrow Q$
- The 3rd sentence is entailed by the first two, but we need an explicit symbol, R, to represent an individual, Confucius, who is a member of the classes *person* and *mortal*
- Representing other individuals requires introducing separate symbols for each, with some way to represent the fact that all individuals who are “people” are also “mortal”

Hunt the Wumpus domain

- Some atomic propositions:

S12 = There is a stench in cell (1,2)

B34 = There is a breeze in cell (3,4)

W22 = Wumpus is in cell (2,2)

V11 = We've visited cell (1,1)

OK11 = Cell (1,1) is safe.

...

- Some rules:

(R1) $\neg S_{11} \rightarrow \neg W_{11} \wedge \neg W_{12} \wedge \neg W_{21}$

(R2) $\neg S_{21} \rightarrow \neg W_{11} \wedge \neg W_{21} \wedge \neg W_{22} \wedge \neg W_{31}$

(R3) $\neg S_{12} \rightarrow \neg W_{11} \wedge \neg W_{12} \wedge \neg W_{22} \wedge \neg W_{13}$

(R4) $S_{12} \rightarrow W_{13} \vee W_{12} \vee W_{22} \vee W_{11}$

...

- The lack of variables requires us to give similar rules for each cell!

1,4	2,4	3,4	4,4
1,3 W!	2,3	3,3	4,3
1,2 A S OK	2,2 OK	3,2	4,2
1,1 V OK	2,1 B V OK	3,1 P!	4,1

A = Agent
 B = Breeze
 G = Glitter, Gold
 OK = Safe square
 P = Pit
 S = Stench
 V = Visited
 W = Wumpus

After the third move

We can prove that the Wumpus is in (1,3) using the four rules given.

See R&N section 7.5

1,4	2,4	3,4	4,4
1,3 W!	2,3	3,3	4,3
1,2 A S OK	2,2 OK	3,2	4,2
1,1 V OK	2,1 B V OK	3,1 P!	4,1

- A** = Agent
- B** = Breeze
- G** = Glitter, Gold
- OK** = Safe square
- P** = Pit
- S** = Stench
- V** = Visited
- W** = Wumpus

Proving W13

Apply MP with $\neg S11$ and R1:

$$\neg W11 \wedge \neg W12 \wedge \neg W21$$

Apply And-Elimination to this, yielding 3 sentences:

$$\neg W11, \neg W12, \neg W21$$

Apply MP to $\sim S21$ and R2, then apply And-elimination:

$$\neg W22, \neg W21, \neg W31$$

Apply MP to S12 and R4 to obtain:

$$W13 \vee W12 \vee W22 \vee W11$$

Apply Unit resolution on $(W13 \vee W12 \vee W22 \vee W11)$ and $\neg W11$:

$$W13 \vee W12 \vee W22$$

Apply Unit Resolution with $(W13 \vee W12 \vee W22)$ and $\neg W22$:

$$W13 \vee W12$$

Apply UR with $(W13 \vee W12)$ and $\neg W12$:

$$W13$$

QED

Propositional Wumpus hunter problems

- Lack of variables prevents stating more general rules
 - We need a set of similar rules for each cell
- Change of the KB over time is difficult to represent
 - Standard technique is to index facts with the time when they're true
 - This means we have a separate KB for every time point

Propositional logic summary

- Inference is the process of deriving new sentences from old
 - **Sound** inference derives true conclusions given true premises
 - **Complete** inference derives all true conclusions from a set of premises
- A **valid sentence** is true in all worlds under all interpretations
- If an implication sentence can be shown to be valid, then—given its premise—its consequent can be derived
- Different logics make different **commitments** about what the world is made of and what kind of beliefs we can have
- **Propositional logic** commits only to the existence of facts that may or may not be the case in the world being represented
 - Simple syntax and semantics suffices to illustrate the process of inference
 - Propositional logic can become impractical, even for very small worlds