ME401 Finite Element Analysis
UNIT I INTRODUCTION

The differential equation of physical phenomenon is given by $\frac{d^2y}{dx^2}+500x^2=0,\ 0\le x\le 1,$

Trial function, $y=a_1(x-x^4)$, Boundary condition are, y(0)=0, y(1)=0 calculate the value of the parameter a_1 by the following methods. (i) Point collocation method (ii) Sub-domain collocation method (iii) least Square Method and (iv) Galerkin's method.

Given: Differential equation $\frac{d^2y}{dx^2}+500x^2=0,\ 0\le x\le 1$

Trial function, $y=a_1(x-x^4)$

Boundary condition are, $y(0)=0, y(1)=0$

To find: The value of parameter a_1 by,
i. point collocation method, ii. Subdomain method, iii. Least Squares method, iv. Galerkin method.

Solution: First we have to verify, whether the trial function satisfies the boundary condition or not.

Trial function is, $y=a_1(x-x^4)$

When $x=0$, $y=a_1(0-0)=0$

$x=1$, $y=a_1(1-1^4)=0$

Hence it satisfies the boundary conditions,

(i) point collocation method:

$y=a_1(x-x^4)$

$dy/dx = a_1(1-4x^3)$

$d^2y/dx^2 = a_1(0-12x^2)$

$$\boxed{d^2y/dx^2 = -12a_1x^2}$$

Substituting $\frac{d^2y}{dx^2}$ value in given differential equation (1), → $\frac{d^2y}{dx^2} + 500x^2 = 0$

$\Rightarrow$ Residual, $R = -12a_1x^2 + 500x^2$ → (2)

In point collocation method, residuals are set to zero.

$R = -12a_1x^2 + 500x^2 = 0$ → (3)

In this problem, we have to find only one parameter, a_1. So, only one collocation point is needed.

The point may be chosen between 0 and 1. Let us take $1/2$,

Substituting $x = \frac{1}{2}$ in equation (3)

$$R = -12\,a_1\left[\frac{1}{2}\right]^2 + 500\left[\frac{1}{2}\right]^2 = 0$$

$$\Rightarrow -\cancel{12}^{\,3}\,a_1\left[\frac{1}{\cancel{4}}\right] + \cancel{500}^{\,125}\left[\frac{1}{\cancel{4}}\right] = 0$$

$$-3a_1 + 125 = 0$$

$$a_1 = 41.66$$

→ (4)

Hence the trial function is $y = 41.66(x - x^4)$

(ii) Subdomain collocation method:

This method requires $\int_0^1 R\,dx = 0$

Substitute R Value

$$\Rightarrow \int_0^1 [-12a_1x^2 + 500x^2]\,dx = 0$$

$$-12a_1\left[\frac{x^3}{3}\right]_0^1 + 500\left[\frac{x^3}{3}\right]_0^1 = 0$$

$$-\frac{12a_1}{3}[1-0] + \frac{500}{3}[1-0] = 0$$

$$-\frac{12a_1}{\cancel{3}} + \frac{500}{\cancel{3}} = 0$$

$$-12a_1 + 500 = 0$$

$$-12a_1 = -500$$

$$a_1 = \frac{500}{12} = 41.66 \quad \rightarrow 5$$

Trial function is $y = 41.66(x - x^4)$

(iii) Least squares method:

This method require, $I = \int_0^1 R^2\,dx$

It can also be written as

$$\frac{\partial I}{\partial a_1} = \int_0^1 R\frac{\partial R}{\partial a_1}\,dx \quad \rightarrow (6)$$

We Know that, $R = -12a_1x^2 + 500x^2$

$$\frac{\partial R}{\partial a_1} = -12x^2$$

Substitute R and $\frac{\partial R}{\partial a_1}$ values in eq ⑥

$$\Rightarrow \frac{\partial I}{\partial a_1} = \int_0^1 [-12a_1x^2 + 500x^2] (-12x^2) dx$$

The requirement is, $\frac{\partial I}{\partial a_1} = 0$

$$\Rightarrow \int_0^1 [-12a_1x^2 + 500x^2)(-12x^2) dx = 0$$

$$\int_0^1 [144a_1x^4 - 6000x^4) dx = 0$$

$$144a_1\left[\frac{x^5}{5}\right]_0^1 - 6000\left[\frac{x^5}{5}\right]_0^1 = 0$$

$$\frac{144a_1}{5}[1-0] - \frac{6000}{5}[1-0] = 0$$

$$\frac{144a_1}{5} - \frac{6000}{5} = 0 \Rightarrow a_1 = \frac{6000}{144} = 41.66$$

$$a_1 = 41.66$$

(iv) Galerkin's method: In this method, the trial function itselfs is considered as the weighting function, $w_i \Rightarrow \int w_i R\,dx = 0$ ... (8)

Here, the trial function is $y = w_i = a_1(x - x^4)$

Substitute w_i and R values in equation (8)

$$\int_0^1 \cancel{a_1}(x - x^4)(-12a_1 x^2 + 500x^2)\,dx = 0$$

$$a_1 \int_0^1 (x - x^4)(-12a_1 x^2 + 500x^2)\,dx = 0$$

$$a_1 \int_0^1 [-12a_1 x^3 + 500x^3 + 12a_1 x^6 - 500x^6]\,dx = 0$$

$$a_1\left[-12a_1\left[\frac{x^4}{4}\right]_0^1 + 500\left[\frac{x^4}{4}\right]_0^1 + 12a_1\left[\frac{x^7}{7}\right]_0^1 - 500\left[\frac{x^7}{7}\right]_0^1\right] = 0$$

$$-\frac{12a_1}{4}[1-0] + \frac{500}{4}[1-0] + \frac{12a_1}{7}(1-0) - \frac{500}{7}(1-0) = 0$$

$$-3a_1 + 125 + 1.714a_1 - 71.428 = 0$$

$$-1.286\,a_1 = -53.572$$

$$a_1 = 41.66 \quad \text{... (9)}$$

Trial function is $y = 41.66(x - x^4)$

From equation 4, 5, 7, and 9, we know that the value of parameter a_1 is same for all the four methods.

Result: parameter, a_1 [For all the four methods] $= 41.66$

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