UNIT-II Vector Calculus

The vector differential operator ∇ is defined as

① $\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$

② $\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\phi$

③ $\text{grad}\,\phi = \nabla\phi$

Problems:

1. Find grad ϕ where $\phi = x^2+y^2+z^2$

Sol

$$\phi = x^2+y^2+z^2$$

$$\text{grad}\,\phi = \nabla\phi$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(x^2+y^2+z^2) + \vec{j}\frac{\partial}{\partial y}(x^2+y^2+z^2) + \vec{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)$$

$$= \vec{i}(2x) + \vec{j}(2y) + \vec{k}(2z)$$

$$= 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

2. Find grad ϕ where $\phi = xyz$ at $(1,1,1)$

Sol

Given $\phi = xyz$

$$\text{grad}\,\phi = \vec{\nabla}\phi$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(xyz) + \vec{j}\frac{\partial}{\partial y}(xyz) + \vec{k}\frac{\partial}{\partial z}(xyz)$$

$$= \vec{i}(yz) + \vec{j}(xz) + \vec{k}(xy)$$

$$(\nabla\phi)_{(1,1,1)} = \vec{i}(1.1) + j(1.1) + \vec{k}(1,1)$$

$$\text{grad}\,\phi = \vec{i} + \vec{j} + \vec{k}$$

(3) Find grad ϕ where $\phi = 3x^2y - y^3z^2$ at $(1,1)$

Given $\phi = 3x^2y - y^3z^2$

$$\text{grad } \phi = \nabla \phi$$

$$\nabla \phi = \vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \vec{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \vec{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= \vec{i}(6xy) + \vec{j}(3x^2 - 3y^2z^2) + \vec{k}(-2zy^3)$$

$$= 6xy\vec{i} + \vec{j}(3x^2 - 3y^2z^2) - 2y^3z\vec{k}$$

$$\nabla\phi_{(1,1,1)} = 6(1)(1)\vec{i} + (3 - 3(1)(1))\vec{j} - \vec{k}\,2(1)(1)$$

$$= 6\vec{i} + (3-3)\vec{j} - 2\vec{k} = 6\vec{i} + 0\vec{j} - 2\vec{k}$$

(4) Find grad ϕ where $\phi = x^2y^2z^2 + 4xz^2 + xy$ at $(1,2,3)$

Sol

Given $\phi = x^2y^2z^2 + 4xz^2 + xy$

$$\text{grad } \phi = \nabla \phi$$

$$\nabla \phi = \vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(x^2y^2z^2 + 4xz^2 + xy) + \vec{j}\frac{\partial}{\partial y}(x^2y^2z^2 + 4xz^2 + xy) + \vec{k}\frac{\partial}{\partial z}(x^2y^2z^2 + 4xz^2 + xy)$$

$$= \vec{i}(2xy^2z^2 + 4z^2 + y) + \vec{j}(2yx^2z^2 + x) + \vec{k}(x^2y^2 2z + 8xz)$$

$$\nabla_{(1,2,3)} = \vec{i}$$

$$= \vec{i}$$

$$= 110\vec{i} + 37\vec{j} + 48\vec{k}$$

UNIT NORMAL VECTOR

① $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$

② $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$|\vec{r}| = \sqrt{x^2+y^2+z^2}$

Problems:

① Find the unit normal to the surface $x^2+y^2+z^2=1$ at $(1,1,1)$

Sol Given $\phi = x^2+y^2+z^2-1$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(x^2+y^2+z^2-1) + \vec{j}\frac{\partial}{\partial y}(x^2+y^2+z^2-1) + \vec{k}\frac{\partial}{\partial z}(x^2+y^2+z^2-1)$$

$$= \vec{i}(2x) + \vec{j}(2y) + \vec{k}(2z)$$

$$= 2x\vec{i} + 2y\vec{j} + 2z\vec{k} \Rightarrow \nabla\phi_{(1,1,1)} = 2\vec{i} + 2\vec{j} + 2\vec{k}$$

$$|\nabla\phi| = \sqrt{2^2+2^2+2^2}$$

$$= \sqrt{4+4+4} = \sqrt{12} = 2\sqrt{3}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2(\vec{i}+\vec{j}+\vec{k})}{2\sqrt{3}} = \frac{\vec{i}+\vec{j}+\vec{k}}{\sqrt{3}}$$

HW

2) Find the unit normal to the surface $x^2-y^2+z^2=2$ at $(1,-1,2)$

Sol Given $\phi = x^2-y^2+z^2-2$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

Directional Derivative

Directional derivative $= \nabla\phi \cdot \dfrac{\vec{a}}{|\vec{a}|}$

$\vec{a} \rightarrow$ direction

NOTE:

$\vec{i}\cdot\vec{i} = 1$ $\quad \vec{i}\times\vec{i} = 0$

$\vec{j}\cdot\vec{j} = 1$ $\quad \vec{j}\times\vec{j} = 0$

$\vec{k}\cdot\vec{k} = 1$ $\quad \vec{k}\times\vec{k} = 0$

Problems:

① Find the directional derivatives of the function x^2+2xy at $(1,-1,3)$ in the direction $\vec{i}+2\vec{j}+2\vec{k}$.

Sol

Given $\phi = x^2 + 2xy$

Directional derivative $= \nabla\phi \dfrac{\vec{a}}{|\vec{a}|}$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(x^2+2xy) + \vec{j}\frac{\partial}{\partial y}(x^2+2xy) + \vec{k}\frac{\partial}{\partial z}(x^2+2xy)$$

$$= \vec{i}(2x+2y) + \vec{j}(2x) + \vec{k}(0)$$

$$\nabla\phi_{(1,-1,3)} = \vec{i}(2(1)+2(-1)) + \vec{j}\,2(1)$$

$$= 2\vec{j}$$

Given $\vec{a} = \vec{i} + 2\vec{j} + 2\vec{k}$

$$|\vec{a}| = \sqrt{1^2+2^2+2^2} = \sqrt{9} = 3$$

directional derivative $= \nabla\phi \dfrac{\vec{a}}{|\vec{a}|}$

$$= 2\vec{j}\left(\frac{\vec{i}+2\vec{j}+2\vec{k}}{3}\right)$$

$$= \frac{2\vec{j}\cdot\vec{i} + 2\vec{j}\cdot 2\vec{j} + 2\vec{j}\cdot 2\vec{k}}{3}$$

$$= \frac{0+4+0}{3} = 4/3$$

② Find the directional derivative of $\phi = xy+yz+zx$ at $(1,2,0)$ in the direction $\vec{i}+2\vec{j}+2\vec{k}$. Find also its maximum value.

Sol

Given $\phi = xy + yz + zx$

directional derivative $= \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}$.

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(xy + yz + zx) + \vec{j}\frac{\partial}{\partial y}(xy + yz + zx) + \vec{k}\frac{\partial}{\partial z}(xy + yz + z$$

$$= \vec{i}(y+z) + \vec{j}(x+z) + \vec{k}(x+y)$$

$$\nabla\phi_{(1,2,0)} = \vec{i}(2+0) + \vec{j}(1+0) + \vec{k}(1+2)$$

$$= 2\vec{i} + \vec{j} + 3\vec{k}$$

Given $\vec{a} = \vec{i} + 2\vec{j} + 2\vec{k}$

$$|\vec{a}| = \sqrt{1^2 + 2^2 + 2^2} = 3$$

directional derivative $= \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}$

$$= 2\vec{i} + \vec{j} + 3\vec{k} \cdot \frac{\vec{i} + 2\vec{j} + 2\vec{k}}{3}$$

$$= \frac{2\vec{i}\cdot\vec{i} + \vec{j}\cdot 2\vec{j} + 3\vec{k}\cdot 2\vec{k}}{3}$$

$$= \frac{2+2+6}{3} = \frac{10}{3}$$

Maximum value $= |\nabla\phi|$

$$\nabla\phi = 2\vec{i} + \vec{j} + 3\vec{k}$$

$$|\nabla\phi| = \sqrt{2^2 + 1^2 + 3^2}$$

$$= \sqrt{4+1+9} = \sqrt{14}$$

NOTE:

1. $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

2. div $\vec{F}$ or $\nabla\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

3. Curl $\vec{F}$ (or) $\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$