



UNIT 1- COMBINATORICS

Solving Linear Recurrence Relation

Linear Non-Homogeneous Recurrence Equations:

A recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + F(n)$$

where c_1, c_2, \dots, c_k are real nos. & $F(n)$ is a fcn. not identically zero depending only on n , is called a non homogeneous recurrence relation with constant coefficients.

Q. Solve the recurrence relation

$$S(k) - 7S(k-1) + 10S(k-2) = 8k + 6 \quad \text{with } S(0) = 1, \\ \rightarrow (1)$$

$$S(1) = 2$$

Given $S(k) - 7S(k-1) + 10S(k-2) = 0$

CE: $m^2 - 7m + 10 = 0$

$$(m-2)(m-5) = 0$$

$$m = 2, 5$$

$$H.S. = A(2)^k + B(5)^k \quad \rightarrow (2)$$

PS

$$RHS = 8k + 6$$

Take $S(k) = ck + d$

$$S(k-1) = c(k-1) + d$$

$$S(k-2) = c(k-2) + d$$

} $\rightarrow (A)$



UNIT 1- COMBINATORICS

Solving Linear Recurrence Relation

Subs. $B=2$ in (5),
 $A+B = -7$
 $A = -2 - 7 = -9$
 $A = -9$

Subs. A & B in (4),
 $S(k) = -9(2)^k + 2(5)^k + 2k + 8$

Q. Solve the Recurrence Relation
 $a_n - a_{n-1} - 6a_{n-2} = -30, a_0 = 0, a_1 = -5, n \geq 2$

Given: $a_n - a_{n-1} - 6a_{n-2} = -30 \rightarrow (1)$

CE: $m^2 - m - 6 = 0$
 $(m-3)(m+2) = 0$
 $m = 3, -2$

HS = $A(3)^n + B(-2)^n \rightarrow (2)$

PS RHS = a constant
Take $a_n = a_{n-1} = a_{n-2} = d$
(1) $\Rightarrow d - d - 6d = -30$
 $-6d = -30$
 $d = 5$
PS = 5 $\rightarrow (3)$

General soln.
 $a_n = A(3)^n + B(-2)^n + 5 \rightarrow (4)$

Given: $a_0 = 0$
 $A + B + 5 = 0$
 $A + B = -5 \rightarrow (5)$

and $a_1 = -5$
 $3A - 2B + 5 = -5$



$3A - 2B = -10 \rightarrow (6)$

Solving (5) and (6),

$$(5) \times 2 \Rightarrow \begin{array}{r} 2A + 2B = -10 \\ 3A - 2B = -10 \\ \hline 5A = -20 \\ A = -\frac{20}{5} = -4 \end{array}$$

(5) $\Rightarrow -4 + B = -5$
 $B = -5 + 4$
 $B = -1$

(4) $\Rightarrow a_n = -4(3)^n - 1(-2)^n + 5$

3]. Solve $a_n - 2a_{n-1} - 3a_{n-2} = 4^n + 6$
 Given $a_n - 2a_{n-1} - 3a_{n-2} = 4^n + 6 \rightarrow (1)$

CE: $m^2 - 2m - 3 = 0$
 $(m-3)(m+1) = 0$
 $m = 3, -1$

HS = $A(3)^n + B(-1)^n \rightarrow (2)$

PS: RHS = $4^n + 6$
 PS = $PS_1 + PS_2$

PS₁: Take $a_n = d \cdot 4^n$
 $a_{n-1} = d \cdot 4^{n-1}$
 $a_{n-2} = d \cdot 4^{n-2} \} \rightarrow (3)$

Subs. (3) in (1),
 $d \cdot 4^n - 2d \cdot 4^{n-1} - 3d \cdot 4^{n-2} = 4^n$
 $d \cdot 4^n - 2d \cdot \frac{4^n}{4} - 3d \cdot \frac{4^n}{16} = 4^n$



UNIT 1- COMBINATORICS

Solving Linear Recurrence Relation

$$4^n \left[d - \frac{d}{2} - \frac{3d}{16} \right] = 4^n$$

$$\frac{16d - 8d - 3d}{16} = 1$$

$$\frac{5d}{16} = 1$$

$$d = \frac{16}{5}$$

$$PS_1 = \frac{16}{5} (4)^n$$

PS₂:

RHS = a constant

Take $a_n = a_{n-1} = a_{n-2} = d$

$$d - 2d - 3d = 6$$

$$-4d = 6$$

$$d = \frac{6}{-4}$$

$$d = \frac{-3}{2}$$

$$PS_2 = \frac{-3}{2}$$

$$PS = \frac{16}{5} (4)^n - \frac{3}{2}$$

General soln.

$$a_n = A(3)^n + B(-1)^n + \frac{16}{5} (4)^n - \frac{3}{2}$$

4]. solve $a_n - 4a_{n-1} + 4a_{n-2} = 2^n + 3n, n \geq 2$

Given. $a_n - 4a_{n-1} + 4a_{n-2} = 2^n + 3n$
 $\hookrightarrow (1)$

CE: $m^2 - 4m + 4 = 0$

$$(m-2)^2 = 0$$

$$m = 2, 2$$

$$HS = (A + nB) 2^n$$



UNIT 1- COMBINATORICS

Solving Linear Recurrence Relation

PS:

$$RHS = 2^n + 3n$$

$$PS = PS_1 + PS_2$$

$PS_1 = 2^n$

Take $a_n = dn^2 2^n$ since base of RHS is 2, which is a double root of CE

$$a_{n-1} = d(n-1)^2 2^{n-1}$$

$$a_{n-2} = d(n-2)^2 2^{n-2}$$

$$\therefore (1) \Rightarrow dn^2 2^n - 4 [d(n-1)^2 2^{n-1}] + 4 [d(n-2)^2 2^{n-2}] = 2^n$$

$$\div 2^n \quad dn^2 - 4d(n-1)^2 2^{-1} + 4d(n-2)^2 2^{-2} = 1$$

$$dn^2 - \frac{4d}{2}(n^2 + 1 - 2n) + \frac{4d}{4}(n^2 + 4 - 4n) = 1$$

$$dn^2 - 2d(n^2 + 1 - 2n) + d(n^2 + 4 - 4n) = 1$$

$$dn^2 - 2dn^2 - 2d + 4dn + dn^2 + 4d - 4dn = 1$$

$$2d = 1$$

$$d = \frac{1}{2}$$

$$PS_1 = \frac{1}{2} n^2 (2)^n$$

$PS_2 = 3n$

Take $a_n = d_0 + d_1 n$

$$a_{n-1} = d_0 + d_1(n-1)$$

$$a_{n-2} = d_0 + d_1(n-2)$$

$$(1) \Rightarrow d_0 + d_1 n - 4 [d_0 + d_1(n-1)] + 4 [d_0 + d_1(n-2)] = 3n$$

$$d_0 + d_1 n - 4d_0 - 4d_1 n + 4d_1 + 4d_0 + 4d_1 n - 8d_1 = 3n$$

$$d_0 - 4d_1 + d_1 n = 3n$$



UNIT 1- COMBINATORICS

Solving Linear Recurrence Relation

Equating the coeff. of n and constant,

$$d_1 = 3; \quad d_0 - 4d_1 = 0$$

$$d_0 = 4d_1 = 12$$

$$d_0 = 12$$

$$P.S_2 = 12 + 3n$$

$$P.S = \frac{1}{2} (n)^2 (2)^n + 12 + 3n$$

General soln.

$$a_n = (A + nB)(2)^n + \frac{1}{2} (n)^2 (2)^n + 12 + 3n$$

Hw J. Solve $S(k) - 5S(k-1) + 6S(k-2) = 2$ with

$$S(0) = 1, \quad S(1) = -1.$$