5]. Solve $a_n - 7a_{n-1} + 10a_{n-2} = 0$, $n \geq 2$ with

$a_0 = 4$, $a_1 = 17$

Givn. $a_n - 7a_{n-1} + 10a_{n-2} = 0$

characteristic eqn: $m^2 - 7m + 10 = 0$

$(m-2)(m-5) = 0$

$m = 2, 5$

HS $\cancel{a_n} = A(2)^n + B(5)^n$

Since RHS $= 0$, PS $= 0$.

The soln. is $a_n = A(2)^n + B(5)^n$

Given $a_0 = 4$

$\Rightarrow a_0 = A(2)^0 + B(5)^0$

$4 = A + B \rightarrow (1)$

and $a_1 = 17 \Rightarrow a_1 = A(2)^1 + B(5)^1$

$17 = 2A + 5B \rightarrow (2)$

Solving (1) and (2),

$A + B = 4$

$2A + 5B = 17$

$(1)\times 2 \Rightarrow 2A + 2B = 8$

$(-) \quad (-) \quad (-)$

$3B = 9 \Rightarrow B = 3$

$(1) \Rightarrow A = 4 - 3$

$A = 1$

$\therefore a_n = 1(2)^n + 3(5)^n$

2]. Solve the recurrence relation $S(K) = -3S(K-1) - 3S(K-2) - S(K-3)$ with the initial conditions $S(0) = 0$, $S(1) = -2$, $S(2) = -1$.

Grn. $S(K) + 3S(K-1) + 3S(K-2) + S(K-3) = 0$

characteristic eqn. $m^3 + 3m^2 + 3m + 1 = 0$

$$\begin{array}{r|rrrr} -1 & 1 & 3 & 3 & 1 \\ & 0 & -1 & -2 & -1 \\ \hline & 1 & 2 & 1 & \boxed{0} \end{array}$$

$m = -1$, $m^2 + 2m + 1 = 0$

$(m+1)^2 = 0$

$m = -1, -1$

$\therefore m = -1, -1, -1$

$HS = (A + Bn + Cn^2)(-1)^n$

Since $RHS = 0 \Rightarrow PC = 0$

$\therefore S(n) = (A + Bn + Cn^2)(-1)^n$

Given $S(0) = 0$ ie., $S(0) = A = 0 \rightarrow (1)$

$S(1) = -2$ ie., $S(1) = (A + B + C)(-1)^1 = -2$

$A + B + C = 2 \rightarrow (2)$

$S(2) = -1 \Rightarrow S(2) = (A + 2B + 4C)(-1)^2 = -1$

$A + 2B + 4C = -1 \rightarrow (3)$

Solving (1), (2) and (3),

Sub. $A = 0$ in (2) & (3),

$$B + C = 2 \rightarrow (4)$$

$$2B + 4C = -1 \rightarrow (5)$$

$$(4) \times 2 \Rightarrow \underset{(-)}{2B} + \underset{(-)}{2C} = \underset{(-)}{4}$$

$$2C = -5$$

$$C = -5/2$$

$$(4) \Rightarrow B = 2 - C = 2 - (-5/2)$$

$$B = 2 + \frac{5}{2}$$

$$B = \frac{9}{2}$$

$$\therefore a_n = \left[\frac{9}{2} n - \frac{5}{2} n^2\right] (-1)^n$$

3] Solve the recurrence relation $a_{n+2} = 4a_{n+1} - 4a_n$, $n \geq 0$, $a_0 = 1$, $a_1 = 3$

Gvn. $a_{n+2} - 4a_{n+1} + 4a_n = 0$ Replace $n \rightarrow n-2$

$$\Rightarrow a_n - 4a_{n-1} + 4a_{n-2} = 0$$

characteristic eqn: $m^2 - 4m + 4 = 0$

$$(m-2)^2 = 0$$

$$m = 2, 2$$

since RHS $= 0$

$$\therefore a_n = (A + Bn)\, 2^n$$

Gvn $a_0 = 1 \Rightarrow A = 1$

$$a_1 = 3 \Rightarrow (A + B)\, 2^1 = 3$$

$$2A + 2B = 3$$

$$2 + 2B = 3 \Rightarrow 2B = 3 - 2 = 1$$

$$B = 1/2$$

$$\therefore a_n = \left(1 + \frac{1}{2} n\right) 2^n$$

4]. Solve the recurrence relation for the fibonacci sequence 0, 1, 1, 2, 3, 5, 8, 13, ...

Soln.:

Fibonacci sequence 0, 1, 1, 2, 3, 5, 8, 13, ...
satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$
with $f_0 = 0$, $f_1 = 1$

ie, $f_n - f_{n-1} - f_{n-2} = 0$

Characteristic eqn.: $m^2 - m - 1 = 0$

$$m = \frac{1 \pm \sqrt{5}}{2}$$

Since RHS = 0 ⇒ PS = 0

∴ The soln. is $f_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$

Given $f_0 = 0 \Rightarrow f_0 = A\left(\frac{1+\sqrt{5}}{2}\right)^0 + B\left(\frac{1-\sqrt{5}}{2}\right)^0$

$$0 = A + B$$

$$A + B = 0 \rightarrow (1)$$

and $f_1 = 1 \Rightarrow f_1 = A\left(\frac{1+\sqrt{5}}{2}\right)^1 + B\left(\frac{1-\sqrt{5}}{2}\right)^1 \rightarrow$

$$A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) = 1 \rightarrow (2)$$

Solving (1) & (2), we get

$$A = \frac{1}{\sqrt{5}} \; ; \; B = \frac{-1}{\sqrt{5}}$$

$$\therefore f_n = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^n - \frac{1}{\sqrt{5}}\left[\frac{1-\sqrt{5}}{2}\right]^n$$

5]. Find the solution to the recurrence relation $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with $a_0 = 2, a_1 = 5$ and $a_2 = 15$.

Given $a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0$

characteristic eqn:

$$m^3 - 6m^2 + 11m - 6 = 0$$

$$\begin{array}{c|cccc} & 1 & -6 & 11 & -6 \\ 1 & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$m = 1$, $m^2 - 5m + 6 = 0$

$(m-2)(m-3) = 0$

$$m = 1, 2, 3$$

Since $RHS = 0 \Rightarrow PS = 0$

$$\therefore a_n = A(1)^n + B(2)^n + C3^n$$

Given $a_0 = 2$

$$A + B + C = 2 \rightarrow (1)$$

$a_1 = 5$

$$A + 2B + 3C = 5 \rightarrow (2)$$

and $a_2 = 15$

$$A + B(2^2) + C(3)^2 = 15$$

$$A + 4B + 9C = 15 \rightarrow (3)$$

Solving (1), (2) and (3),

$$(1) \Rightarrow C = 2 - A - B \rightarrow (4)$$

Sub. (4) in (2),

$$A + 2B + 3(2 - A - B) = 5$$

$$A + 2B + 6 - 3A - 3B = 5$$

$$-2A - B = 5 - 6 = -1$$

$$2A + B = 1 \rightarrow (5)$$

Sub. (4) in (3),

$$A + 4B + 9(2 - A - B) = 15$$

$$A + 4B + 18 - 9A - 9B = 15$$

$$-8A - 5B = -3$$

$$8A + 5B = 3 \rightarrow (6)$$

Solving (5) & (6),

$$\cancel{(5)\times 4 \Rightarrow 8A + 4B = 4}$$

$$\cancel{(6) \Rightarrow 8A +}$$

$$(5)\times 5 \Rightarrow 10A + 5B = 5$$

$$(6) \Rightarrow \underset{(-)}{8A} + \underset{(-)}{5B} = \underset{(-)}{3}$$

$$2A = 2 \Rightarrow A = 1$$

Sub $A=1$ in (5),

$$2+B=1 \Rightarrow B=-1$$

$$(1) \Rightarrow A+B+C=2$$
$$1-1+C=2$$
$$C=2$$

$\therefore$ solution is $a_n = 1(1)^n - 1(2)^n + 2(3)^n$

$$a_n = 1^n - 2^n + 2(3)^n$$

6]. Solve the recurrence relation $a_n = 2a_{n-1} - 2a_{n-2}$, $n \geq 2$ and $a_0=1$, $a_1=2$

Givn.

$$a_n - 2a_{n-1} + 2a_{n-2} = 0$$

characteristic eqn.

$$m^2 - 2m + 2 = 0$$

$$m = \frac{2 \pm \sqrt{4-4(1)(2)}}{2(1)} = \frac{2 \pm \sqrt{4-8}}{2}$$

$$= \frac{2 \pm 2i}{2}$$

$$= \frac{2[1 \pm i]}{2}$$

$$m = 1 \pm i \quad (\alpha \pm i\beta)$$

$\therefore$ solution is $a_n = r^n (A\cos n\theta + B\sin n\theta)$

where $r = \sqrt{\alpha^2+\beta^2}$ and $\theta = \tan^{-1}(\beta/\alpha)$

$r = \sqrt{2}$ $= \tan^{-1}(1/1) = \tan^{-1}(1)$

$$\theta = \pi/4$$

$$\therefore a_n = (\sqrt{2})^n \left[A\cos\frac{n\pi}{4} + B\sin\frac{n\pi}{4}\right] \rightarrow (A)$$

Givn $a_0 = 1 \Rightarrow a_0 = A = 1$

and $a_1 = 2 \Rightarrow a_1 = (\sqrt{2})\left[A\cos\frac{\pi}{4} + B\sin\frac{\pi}{4}\right] = 2$

$$A\sqrt{2} \times \frac{1}{\sqrt{2}} + B\sqrt{2} \times \frac{1}{\sqrt{2}} = 2$$

$$A + B = 2$$

$$B = 2 - 1$$

$$B = 1$$

$\therefore$ Solution is $a_n = (\sqrt{2})^n \left(\cos \frac{n\pi}{4} + i \sin \frac{n\pi}{4} \right)$

HW J. Solve the recurrence relation

$S(n) - 10S(n-1) + 9S(n-2) = 0$ with $S(0) = 3, S(1) = 11$