Find the volume of the sphere $x^2+y^2+z^2=a^2$ without transformation.

Sol

Volume = 8 × Volume of the 1st octant

Given $x^2+y^2+z^2=a^2$

$$x^2+y^2+z^2=a^2$$
$$z^2=a^2-x^2-y^2$$
$$=\pm\sqrt{a^2-x^2-y^2}$$

It takes only +ve terms because it is in I quadrant.

$$\therefore z=\sqrt{a^2-x^2-y^2}$$

After getting z limit at the next limit to become zero

$$x^2+y^2=a^2$$
$$y^2=a^2-x^2$$
$$y=\pm\sqrt{a^2-x^2}$$

It takes only positive term because it is in first quadrant

$$y=\sqrt{a^2-x^2}$$

After getting z, y limit at the next limit for z & y becomes zero

$$x^2=a^2$$
$$x=a$$

Volume of sphere = 8 x volume of 1st octant

$$=8\times\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} dx\,dy\,dz$$

$$=8\times\int_0^a\int_0^{\sqrt{a^2-x^2}}[x]_0^{\sqrt{a^2-x^2-y^2}}\,dy\,dx$$

$$=8\times\int_0^a\int_0^{\sqrt{a^2-x^2}}\sqrt{a^2-x^2-y^2}\,dy\,dx$$

$$=8\times\int_0^a\int_0^{\sqrt{a^2-x^2}}\sqrt{(\sqrt{a^2-x^2})^2-y^2}\,dy\,dx$$

$$\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$$

Ult Term

$$\text{Volume} = 8 \times \int_0^a \left[\frac{y}{2}\sqrt{a^2-x^2-y^2} + \frac{a^2-x^2}{2}\sin^{-1}\frac{y}{\sqrt{a^2-x^2}} \right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 8 \times \int_0^a \left(\frac{\sqrt{a^2-x^2}}{2}\sqrt{a^2-x^2-a^2+x^2} + \frac{a^2-x^2}{2}\sin^{-1}\frac{\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}} \right) dx$$

$$= 8 \times \int_0^a \left[0 + \frac{a^2-x^2}{2}\sin^{-1}(1) \right] dx$$

$$= 8 \times \int_0^a \frac{a^2-x^2}{2} \times \pi/2 \, dx$$

$$= \frac{8\pi}{4} \int_0^a (a^2-x^2)\, dx$$

$$= \frac{8\pi}{4} \left[a^2x - \frac{x^3}{3} \right]_0^a$$

$$= 2\pi \left[\frac{3a^3 - a^3}{3} \right]$$

$$= \frac{4}{3}\pi a^3 \text{ Cubic}$$