TRIPLE INTEGRATION IN CARTESIAN COORDINATES

Triple integration of a function defined over a region $\iiint f(x,y,z)\,dx\,dy\,dz$

Note:

$\iiint_R dx\,dy\,dz \rightarrow$ volume of the region.

Problems:

1. Evaluate $\int_0^3\int_0^2\int_0^1 (x+y+z)\, dz\,dy\,dx$

Sol

$$\int_0^3\int_0^2\int_0^1 (x+y+z)\,dz\,dy\,dx = \int_0^3\int_0^2 \left[xz+yz+\frac{z^2}{2}\right]_0^1 dy\,dx$$

$$= \int_0^3\int_0^2 \left(x+y+\frac{1}{2}\right) dy\,dx$$

$$= \int_0^3 \left(xy+\frac{y^2}{2}+\frac{y}{2}\right)_0^2 dx$$

$$= \int_0^3 (2x+2+1)\,dx$$

$$= \left[\frac{2x^2}{2}+2x+x\right]_0^3$$

$$= 9+6+3$$

$$= 18$$

2. Evaluate $\int_0^1\int_0^2\int_0^3 xyz\; dx\,dy\,dz$

Sol

$$= \int_0^1\int_0^2 \left[\frac{x^2}{2}\right]_0^3 dy\,dz$$

$$= \int_0^1 \frac{9}{2}\int_0^2 yz\,dy\;dz$$

$$= \frac{9}{2}\int_0^1 \left[\frac{y^2}{2}\right]_0^2 z\,dz$$

$$= 9\left[\frac{z^2}{2}\right]_0^1$$

$$= 9/2$$

3. Evaluate $\int_0^a\int_0^b\int_0^c e^{x+y+z}\; dx\,dy\,dz$

Sol

$$= \int_0^a\int_0^b\int_0^c e^x e^y e^z\, dx\,dy\,dz$$

$$= \int_0^a\int_0^b \left[e^x\right]_0^c e^y dy\; e^z\,dz$$

$$= \int_0^a\int_0^b (e^c-e^0)\, e^y e^z\,dy\,dz$$

$$= e^c-1 \int_0^a \left[e^y\right]_0^b e^z\,dz$$

$$= (e^c - 1) \int_0^a (e^b - e^0)\, e^z \, dz$$

$$= (e^c - 1)(e^b - 1) \left[e^z \right]_0^a$$

$$= (e^c - 1)(e^b - 1)(e^a - 1)$$

$= (e^c - 1) \int_0^a (e^b - e^0)\, e^z\, dz$

$= (e^c - 1)(e^b - 1) \left[e^z\right]_0^a$

$= (e^c - 1)(e^b - 1)(e^a - 1)$

HW

1. Evaluate $\int_0^1 \int_0^y \int_0^{x+y} dx\, dy\, dz$

Sol

Given integration is not correct form

$\int_0^1 \int_0^y \int_0^{x+y} dz\, dx\, dy$

$= \int_0^1 \int_0^y \left[z\right]_0^{x+y} dx\, dy$

$= \int_0^1 \int_0^y (x+y)\, dx\, dy$

$= \int_0^1 \left[\frac{x^2}{2} + yx\right]_0^y dy$

$= \int_0^1 \left[\frac{y^2}{2} + y^2\right] dy$

$= \int_0^1 \left(\frac{3y^2}{2}\right) dy$

$= 3/2 \left[\frac{y^3}{3}\right]_0^1$

$= 1/2$