TWO MARKS Q & A

1. Define Cross – Correlation function and state any two of its properties

Ans : The cross correlation of the two random processes $X(t)$ and $Y(t)$ at t_1, t_2 is defined by

$$R_{XY}(t_1, t_2) = E[X(t_1), Y(t_2)]$$

where $X(t_1)$ and $Y(t_2)$ are random variables

Properties :

(i) The cross - correlation function is not generally an even function of τ, but it has a symmetry relationship of

$$R_{XY}(\tau) = R_{YX}(-\tau)$$

(iii) If $X(t)$ $\dot{Y}(t)$ are two random processes and $R_{XX}(\tau)$ and $R_{YY}(\tau)$ are their respective auto correlation functions, then the relation

$$|R_{XY}(\tau)| \leq \sqrt{[R_{XX}(0) + R_{YY}(0)]} \text{ holds good.}$$

2. Define spectral density.

Ans : Let $\{X(t), t \geq 0)$ be a stationery time series with $E[X(t)] = 0$ and covariance function $R(t-s) = E[X(t)X(s)]$ and let $F(x)$ be a real, never decreasing and bounded function of x with $dF(x) = f(x)\,dx$.

R (t) is non-negative definite then

$$F(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(\tau)\, e^{-i\omega\tau}\, d\tau$$

which is called spectral density.

3. Define cross-spectral density.

Ans : Cross spectral density of the two jointly WSS continuous-time process $\{X(t), Y(t)\}$ is defined as the Fourier transform of the Cross-Correlation function $R_{xy}(z)$ given by

$$S_{XY}(f) = \int_{-\infty}^{\infty} R_{XY}(z)\, e^{-i2\pi fz}\, dz$$

4. Prove that $R_{XX}(\tau) = R_{XX}(-\tau)$

Proof : By definition

$$R_{XX}(\tau) = E[X(t)X(t+\tau)] \qquad \ldots (1)$$

Replacing τ by $-\tau$ in (1), we have,

$$R_{XX}(-\tau) = E[X(t)X(t-\tau)]$$

Put

$$t-\tau = t_1$$

$$\Rightarrow \quad t = t_1 + \tau.$$

$$\therefore R_{XX}(-\tau) = E[X(t_1+\tau)X(t_1)]$$

$$= E[X(t_1)X(t_1+\tau)]$$

$$= R_{XX}(\tau) \qquad \text{[by (1)]}$$

Hence $\mathbf{R_{XX}(-\tau) = R_{XX}(\tau)}$.

Hence the property is proved.

5. **Check whether the following functions are valid auto correlation function $\frac{1}{1+9\tau^2}$.**

Ans : Given

$$R_{XX}(\tau) = \frac{1}{1+9\tau^2}$$

$$R_{XX}(-\tau) = \frac{1}{1+9(-\tau)^2}$$

$$= \frac{1}{1+9\tau^2} = R_{XX}(\tau)$$

Also $R_{XX}(0) = 1$

For all τ, $R_{XX}(\tau) < R_{XX}(0)$.

$\therefore$ The given function is an auto correlation function.

6. **Check whether the following functions are valid Autocorrelation functions $R_{XX}(\tau) = \cos(\tau) + \frac{|\tau|}{T}$.**

Ans :

$$R_{XX}(\tau) = \cos(\tau) + \frac{|\tau|}{T}$$

$$R_{XX}(-\tau) = \cos(-\tau) + \frac{|-\tau|}{T} = \cos\tau + \frac{|\tau|}{T}$$

$$= R_{XX}(\tau)$$

Hence it is a valid Autocorrelation function.

7. **Find the mean and variance of a stationary random process whose auto correlation function is given by $R_{XX}(\tau) = 18 + \frac{2}{6+\tau^2}$.**

[A.U. May '06, Jan '07, Nov '08]

Ans : Given

$$R_{XX}(\tau) = 18 + \frac{2}{6+\tau^2}$$

Since the Random process is stationary, by *Property 5* of Auto correlation function we have,

$$\overline{X}^2 = \lim_{|\tau| \to \infty} R_{XX}(\tau)$$

$$\therefore \overline{X}^2 = \lim_{|\tau| \to \infty} \left[18 + \frac{2}{6+\tau^2}\right] \quad \dots (1)$$

$$= 18 + \lim_{|\tau| \to \infty} \frac{2}{6+\tau^2} = 18 + \frac{2}{6+\infty}$$

$$= 18 + 0 = 18 \quad \left[\because \frac{x}{\infty} = 0\right]$$

$$\therefore E[X(t)] = \overline{X} = \sqrt{18} = 3\sqrt{2}.$$

Also $\text{Var}\{X(t)\} = E[X^2(t)] - \{E[X(t)]\}^2$

But, we know that

$$E[X^2(t)] = R_{XX}(0)$$

$$= 18 + \frac{2}{6+0} = \frac{55}{3}$$

$$\therefore \text{Variance of } \{X(t)\} = \text{Var}\{X(t)\}$$

$$= E[X^2(t)] - \{E[X(t)]\}^2$$

$$= \frac{55}{3} - 18$$

$$= \frac{55-54}{3} = \frac{1}{3}$$

8. Determine the mean and variance of the process given that the auto correlation function $R_{XX}(\tau) = 25 + \frac{4}{1+6\tau^2}$. Find the mean value and variance of the process $\{X(t)\}$.

Ans : $R_{XX}(\tau) = 25 + \frac{4}{1+6\tau^2}$

Let us assume that the process be stationary. Then we know that

$$\overline{X}^2 = \lim_{|\tau| \to \infty} R_{XX}(\tau) \quad \dots (1)$$

Substituting $R_{XX}(\tau)$ in (1), we have,

$$\overline{X}^2 = \lim_{|\tau| \to \infty} \left[25 + \frac{4}{1+6\tau^2}\right]$$

$$= 25 + 0 = 25.$$

$$\therefore \overline{X}^2 = 25$$

$$\Rightarrow \quad \overline{X} = 5$$

$\therefore$ Mean = E (X (t)) = 5 ... (1) (a)

The Variance is defined by

$$\sigma^2 = E[X^2(t)] - [E[X(t)]]^2 \quad \ldots (2)$$

We know that,

$$E[X^2(t)] = R_{XX}(0).$$

$$= 25 + \frac{4}{1+6(0)}$$

$$= 25 + 4 = 29. \quad \ldots (3)$$

Substituting (3) and (1) (a) in (2), we get

$$\text{Variance } \sigma^2 = 29 - 5^2$$

$$= 29 - 25$$

$$\sigma^2 = 4$$

9. **Find the mean Square Value of the random process whose Auto Correlation is $\frac{A^2}{2}$ cos ωτ.** **[A.U. Jan '06, Apr '07, Nov '08]**

Ans: Given the auto correlation

$$R_{XX}(\tau) = \frac{A^2}{2} \cos \omega\tau.$$

$$\text{Mean Square value} = R_{XX}(0)$$

$$= \frac{A^2}{2} \cos 0$$

$$= \frac{A^2}{2} \qquad [\because \cos 0 = 1]$$

$$\therefore \text{Mean Square Value} = \frac{A^2}{2}$$

10. **Prove that $R_{XY}(\tau) = R_{YX}(-\tau)$**

Proof :

$$R_{XY}(\tau) = E[X(t)\, Y(t+\tau)] \quad \ldots (1)$$

$$R_{YX}(\tau) = E[Y(t)\, X(t+\tau)] \quad \ldots (2)$$

Replacing τ by – τ in (2), we have

$$R_{YX}(-\tau) = E[X(t)\, Y(t-\tau)] \quad \ldots (3)$$

Put

$$t - \tau = z$$

Then (3) becomes

$$R_{YX}(-\tau) = E[Y(z+\tau)\, X(z)]$$

$$= E[X(z)\, Y(z+\tau)] \quad \text{(By (1))}$$

$$= R_{XY}(\tau)$$

11. Is $\frac{\omega^2+4}{4\omega^4+3\omega^2+5}$ a valid power density spectrum?

Ans : Let $S_{XX}(\omega) = \frac{\omega^2+4}{4\omega^4+3\omega^2+5}$

$$S_{XX}(-\omega) = \frac{(-\omega)^2+4}{4(-\omega)^4+3(-\omega)^2+5} = \frac{\omega^2+4}{4\omega^4+3\omega^2+5}$$

$$= S_{XX}(\omega)$$

Since $S_{XX}(\omega) = S_{XX}(-\omega)$

∴ The given function is a valid power spectrum.

12. The power spectral density of a random process {X (t)} is given by $S_{XX}(\omega) = \begin{cases} \pi, & |\omega| < 1 \\ 0, & \text{elsewhere} \end{cases}$ Find its auto correlation function.

Ans : $R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{i\omega\tau}\, d\omega$

$$= \frac{1}{2\pi}\int_{-1}^{1} \pi e^{i\omega\tau}\, d\omega$$

$$= \frac{1}{2}\left[\frac{e^{i\omega\tau}}{i\tau}\right]_{-1}^{1} = \frac{1}{2}\left[\frac{e^{i\tau}-e^{-i\tau}}{i\tau}\right]$$

$$= \frac{1}{\tau}\sin\tau$$

13. The auto correlation of a stationary random process is given by $R_{XX}(\tau) = ae^{-b|\tau|}$, $b > 0$. Find the spectral density function.

[A.U., Dec. '03]

Ans : Given the Auto Correlation function

$$R_{XX}(\tau) = ae^{-b|\tau|} \quad \text{... (1)}$$

The spectral density function is given by

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} ae^{-b|\tau|}\, e^{-i\omega\tau}\, d\tau \quad \text{... (2)}$$

In $(-\infty, 0)$, $|\tau| = -\tau$

$(0, \infty)$, $|\tau| = \tau$

∴ Equation (2) becomes

$$S_{XX}(\omega) = \int_{-\infty}^{0} ae^{-b(-\tau)}\, e^{-i\omega\tau}\, d\tau + \int_{0}^{\infty} ae^{-b\tau}\, e^{-i\omega\tau}\, d\tau$$

$$= a\int_{-\infty}^{0} e^{b\tau}\, e^{-i\omega\tau}\, d\tau + a\int_{0}^{\infty} e^{-b\tau}\, e^{-i\omega\tau}\, d\tau$$

$$= a\int_{-\infty}^{0} e^{(b-i\omega)\tau}\, d\tau + a\int_{0}^{\infty} e^{-(b+i\omega)\tau}\, d\tau$$

$$= a\left[\frac{e^{(b-i\omega)\tau}}{b-i\omega}\right]_{-\infty}^{0} + a\left[\frac{e^{-(b+i\omega)\tau}}{-(b+i\omega)}\right]_{0}^{\infty}$$

$$= \frac{a}{b-i\omega}[e^{0}-e^{-\infty}] + \frac{a}{-(b+i\omega)}[e^{-\infty}-e^{-0}]$$

$$= \frac{a}{b-i\omega} - \frac{a}{b+i\omega}[0-1] \qquad [\ldots e^{-\infty}=0]$$

$$= \frac{a}{b-i\omega} + \frac{a}{b+i\omega} = a\left[\frac{1}{b-i\omega}+\frac{1}{b+i\omega}\right]$$

$$= a\left[\frac{b+i\omega+b-i\omega}{b^2+\omega^2}\right] = \frac{2ab}{b^2+\omega^2}$$

14. **Define auto correlation function and prove that for a WSS process $\{X(t)\}$, $R_{XX}(-\tau) = R_{XX}(\tau)$**

Ans: The auto correlation $R_{XX}(t_1, t_2)$ of the random process $X(t)$ is defined as the expected value of the product $X(t_1)$ and $X(t_2)$

$$R_{XX}(t_1, t_2) = E[X(t_1)\, X(t_2)]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1\, x_2\, f(x_1, x_2, t_1, t_2)\, dx_1\, dx_2$$

To prove $R_{XX}(\tau) = R_{XX}(-\tau)$

We know that,

$$R_{XX}(t, t+\tau) = R_{XX}(\tau)$$

$$R_{XX}(\tau) = E[X(t) \cdot X(t+\tau)]$$

Replacing τ by $-\tau$, we get

$$R_{XX}(-\tau) = E[X(t), X(t-\tau)] \qquad \ldots (i)$$

Putting $t - \tau = K$ in the R.H.S. of (i) we get

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$$R_{XX}(-\tau) = E[X(K+\tau) \cdot X(K)]$$
$$= E[X(K) \cdot (K+\tau)]$$
$$= R_{XX}(\tau)$$

$\therefore \quad R_{XX}(-\tau) = R_{XX}(\tau)$

15. State any two properties of an autocorrelation function.

Ans : Autocorrelation function is even.

(1) $R_{XX}(\tau) = R_{XX}(-\tau)$

(2) $|R_{XX}(\tau)| = R_{XX}(0)$

16. The power spectral density of a random process X (*t*) is given by $S_{XX}(\omega) = \begin{cases} \pi & \text{if } |\omega| < 1 \\ 0 & \text{elsewhere} \end{cases}$ **. Find its autocorrelation function.**

Ans :
$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{i\omega\tau}\, d\tau$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \pi \cdot e^{i\omega\tau}\, d\tau$$
$$= \frac{1}{2}\left[\frac{e^{i\omega\tau}}{i\tau}\right]_{-1}^{1} = \frac{e^{i\tau} - e^{-i\tau}}{2i\tau}$$
$$= \frac{1}{\tau} \sin\tau$$