Convolution:

Defn:

If $f(t)$ and $g(t)$ are two fns defined for $t \geq 0$ then the Convolution of $f(t)$ and $g(t)$ is defined as

$$f(t) * g(t) = (f * g)(t) = \int_0^t f(u)\, g(t-u)\, du$$

Note:

$$f(t) * g(t) = g(t) * f(t)$$

Convolution theorem:

If $f(t)$ and $g(t)$ are two Laplace transformable fns defined for $t \geq 0$ then $L[f(t) * g(t)]$ is given by,

$$L[f(t) * g(t)] = L[f(t)] * L[g(t)]$$

$$L^{-1}[F(s) \cdot G(s)] = L^{-1}[F(s)] * L^{-1}[G(s)]$$

Problems

① Using convolution theorem, find the inverse transform of

(i) $\dfrac{s^2}{(s^2+a^2)^2}$ (ii) $\dfrac{s}{(s^2+a^2)(s^2+b^2)}$ (iii) $\dfrac{1}{(s+a)(s+b)}$

Solution:

(i) $L^{-1}\left[\dfrac{s^2}{(s^2+a^2)^2}\right] = L^{-1}\left[\dfrac{s}{s^2+a^2}\cdot\dfrac{s}{s^2+a^2}\right]$

$$= L^{-1}\left(\frac{s}{s^2+a^2}\right) * L^{-1}\left(\frac{s}{s^2+a^2}\right)$$

$$= \cos at * \cos at$$

$$= \int_0^t \cos au \cos a(t-u)\, du$$

$$= \int_0^t \left[\frac{\cos(\cancel{au}+at-\cancel{au}) + \cos(au-at+au)}{2}\right] du$$

$$\boxed{\because \cos A \cos B = \frac{\cos(A+B)+\cos(A-B)}{2}}$$

$$= \frac{1}{2}\int_0^t [\cos at + \cos(2au-at)]\, du$$

$$= \frac{1}{2}\left[\cos at \cdot u + \frac{\sin(2au-at)}{2a}\right]_0^t$$

$$= \frac{1}{2}\left[\cos at \cdot t + \frac{\sin(2at-at)}{2a} - \frac{\sin(0-at)}{2a}\right]$$

$$= \frac{1}{2}\left[t\cos at + \frac{\sin at}{2a} + \frac{\sin at}{2a}\right]$$

$$= \frac{1}{2}\left[t\cos at + \frac{\sin at}{a}\right]$$

$$= \frac{1}{2a}(at\cos at + \sin at)$$

(ii) $L^{-1}\left[\frac{s}{(s^2+a^2)(s^2+b^2)}\right]$

$$= L^{-1}\left[\frac{s}{s^2+a^2} \cdot \frac{1}{s^2+b^2}\right]$$

$$= L^{-1}\left(\frac{s}{s^2+a^2}\right) * L^{-1}\left(\frac{1}{s^2+b^2}\right)$$

$$= \cos at * \frac{1}{b} L^{-1}\left(\frac{b}{s^2+b^2}\right)$$

$$= \cos at * \frac{1}{b}\sin bt$$

$$= \frac{1}{b}\int_0^t \cos au \sin b(t-u)\, du$$

$$= \frac{1}{b}\int_0^t \frac{\sin(bt-bu+au) + \sin(bt-bu-au)}{2}\, du$$

$$\boxed{\because \sin A \cos B = \frac{\sin(A+B)+\sin(A-B)}{2}}$$

$$= \frac{1}{2b}\int_0^t \sin[bt+(a-b)u] + \sin[bt-(a+b)u]\, du$$

$$= \frac{1}{2b}\left\{\frac{-\cos[bt+(a-b)u]}{a-b} - \frac{\cos[bt-(a+b)u]}{-(a+b)}\right\}_0^t$$

$$= \frac{1}{2b}\left\{\frac{-\cos(\cancel{bt}+at-\cancel{bt})}{a-b} + \frac{\cos bt}{a-b} + \frac{\cos(\cancel{bt}-at-\cancel{bt})}{a+b} - \frac{\cos bt}{a+b}\right\}$$

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$$= \frac{1}{2b}\left\{\frac{-\cos at}{a-b} + \frac{\cos bt}{a-b} + \frac{\cos at}{a+b} - \frac{\cos bt}{a+b}\right\}$$

$$= \frac{1}{2b}\left\{\frac{-1}{a-b}\left[\cos at - \cos bt\right] + \frac{1}{a+b}\left[\cos at - \cos bt\right]\right\}$$

$$= \frac{1}{2b}(\cos at - \cos bt)\left\{\frac{-1}{a-b} + \frac{1}{a+b}\right\}$$

$$= \frac{1}{2b}(\cos at - \cos bt)\left[\frac{-(\cancel{a}+b)+(\cancel{a}-b)}{a^2-b^2}\right]$$

$$= \frac{1}{\cancel{2b}}(\cos at - \cos bt)\frac{(-\cancel{2b})}{a^2-b^2}$$

$$= \frac{\cos bt - \cos at}{a^2-b^2}$$

(iii) $$L^{-1}\left[\frac{1}{(s+a)(s+b)}\right] = L^{-1}\left(\frac{1}{s+a}\cdot\frac{1}{s+b}\right)$$

$$= L^{-1}\left(\frac{1}{s+a}\right) * L^{-1}\left(\frac{1}{s+b}\right)$$

$$= e^{-at} * e^{-bt}$$

$$= \int_0^t e^{-au}\, e^{-b(t-u)}\, du$$

$$= \int_0^t e^{-au-bt+bu}\, du$$

$$= \left[\frac{e^{-au-bt+bu}}{-a+b}\right]_0^t$$

$$= \left[\frac{e^{-at-\cancel{bt}+\cancel{bt}}}{-a+b} - \frac{e^{-bt}}{-a+b}\right]$$

$$= \frac{e^{-at} - e^{-bt}}{b-a}$$