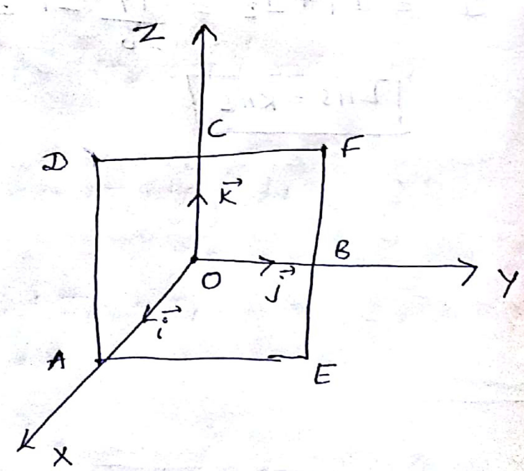
Gauss divergence theorem

If $\vec{F}$ is a vector point function, finite and differentiable in a region R bounded by a closed surface S then the surface integral of the normal component of $\vec{F}$ taken over S is equal to the integral of divergence of $\vec{F}$ taken over V.

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dv$$

(or) $$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint \text{div}\,\vec{F}\, dx\, dy\, dz$$



Face	Unit normal vector $\hat{n}$	ds
S_1 $(x=a)$ AEGD	$\vec{i}$	dy dz
S_2: $(x=0)$ OBFC	$-\vec{i}$	dy dz
S_3: $(y=b)$ EBFG	$\vec{j}$	dx dz
S_4: $(y=0)$ OADC	$-\vec{j}$	dx dz
S_5: $(z=c)$ DGFC	$\vec{k}$	dx dy
S_6: $(z=0)$ OAEB	$-\vec{k}$	dx dy

(1) Verify divergence theorem for $\vec{F} = xy^2\vec{i} + yz^2\vec{j} + zx^2\vec{k}$ over the region bounded by $x=0$, $x=1$, $y=0$, $y=2$, $z=0$ and $z=3$.

Soln:

~~RHS~~ By Gauss divergence theorem,

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V (\nabla\cdot\vec{F})\,dx\,dy\,dz$$

RHS: $\nabla\cdot\vec{F} = y^2 + z^2 + x^2$

$$\iiint_V (\nabla\cdot\vec{F})\,dx\,dy\,dz = \int_0^3\int_0^2\int_0^1 (y^2+z^2+x^2)\,dx\,dy\,dz$$

$$= \int_0^3\int_0^2 \left(y^2x + z^2x + \frac{x^3}{3}\right)_0^1 dy\,dz$$

$$= \int_0^3\int_0^2 \left(y^2 + z^2 + \frac{1}{3}\right) dy\,dz$$

$$= \int_0^3 \left[\frac{y^3}{3} + z^2y + \frac{y}{3}\right]_0^2 dz$$

$$= \int_0^3 \left[\frac{8}{3} + 2z^2 + \frac{2}{3}\right] dz$$

$$= \left[\frac{8}{3}z + \frac{2z^3}{3} + \frac{2}{3}z\right]_0^3$$

$$= \frac{8}{3}\times 3 + 2\times\frac{3^3}{3} + \frac{2}{3}\times 3$$

$$= 8 + 18 + 2$$

$\boxed{RHS = 28}$