2) Verify Green's Theorem in the XY plane for $\int_C \{(3x^2-8y^2)dx+(4y-6xy)dy\}$, where C is the boundary of the region given by $x=y^2$ and $y=x^2$.

Solution:

Using Green's Theorem, we have,

$$\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \quad (1)$$

RHS

~~on~~ comparing, the given integral with LHS of (1) we have

$M = (3x^2 - 8y^2)$ and

$N = (4y - 6xy)$

Now we have to find,

$$\frac{\partial M}{\partial y} = -16y$$

$$\frac{\partial N}{\partial x} = -6y$$

$$\therefore \int_C (3x^2-8y^2)dx+(4y-6xy)dy$$

$$= \iint_R (-6y-(-16y))\,dx\,dy$$

$$= \iint_R 10y\,dx\,dy$$

$$= 10\iint_R y\,dx\,dy$$

To find the Region R.

Given regions are

$x = y^2$ (2) and $y = x^2$ (3).

on solving (2) and (3)

sub (2) in (3),

$y = (y^2)^2 \Rightarrow y = y^4$

$y^4 - y = 0$

$y(y^3-1) = 0$

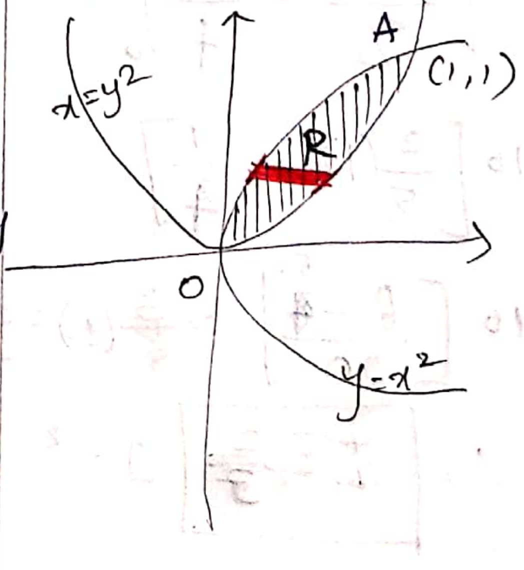
$y=0$ and $y^3=1$

$y=1$

when $y=0$; $x=0$.

when $y=1$; $x=1$

$\therefore$ The point of intersection is $(1,1)$ and $(0,0)$.



∴ The limits for R are

x limits : $x = y^2$ to $x = \sqrt{y}$

(horizontal strip)

y limits : $y = 0$ to $y = 1$

∴ Given Integral

$$= 10\int_0^1 \int_{y^2}^{\sqrt{y}} y \, dx \, dy$$

$$= 10\int_0^1 \left[yx \right]_{x=y^2}^{x=\sqrt{y}} dy$$

$$= 10\int_0^1 y\left[\sqrt{y} - y^2\right] dy$$

$$= 10\int_0^1 \left(y^{3/2} - y^3\right) dy$$

$$= 10\left[\frac{y^{3/2+1}}{3/2+1} - \frac{y^4}{4}\right]_0^1$$

$$= 10\left[\frac{y^{5/2}}{5/2} - \frac{y^4}{4}\right]_0^1$$

$$= 10\left[\frac{2}{5}(1) - \frac{1}{4}\right]$$

$$= 10\left[\frac{8-5}{20}\right]$$

$$\boxed{I = \frac{3}{2}}$$

For LHS

$$\int_C M\,dx + N\,dy$$

$$= \int_{OA} + \int_{AO}$$

Along OA

$$x^2 = y$$

$$\Rightarrow 2x\,dx = dy$$

$$\Rightarrow dy = 2x\,dx$$

$$\int_{OA} = \int_0^1 (3x^2 - 8x^4)\,dx + (4x^2 - 6x^3)\,2x\,dx$$

$$\cancel{= \int_0^1 \left[6x - 32x + 8(3x^2) - 12(4x^3)\right]_0^1}$$

$$= \int_0^1 \left(3x^2 - 8x^4 + 8x^3 - 12x^4\right) dx$$

$$= \left[\frac{3x^3}{3} - \frac{8x^5}{5} + \frac{8x^4}{4} - \frac{12x^5}{5}\right]_0^1$$

$$= \left[x^3 - \frac{8x^5}{5} + 2x^4 - \frac{12}{5}x^5\right]_0^1$$

$$= 1 - \frac{8}{5} + 2 - \frac{12}{5}$$

$$= 3 - \frac{20}{5} = \frac{15-20}{5}$$

$$= \frac{-5}{5} = -1 \;//$$

Along AO

$$\int_{AO} = \int_1^0 (3y^4 - 8y^2)\, 2y\,dy + (4y - 6y^3)\,dy$$

$$= \int_1^0 6y^5 - 16y^3 + 4y - 6y^3\, dy$$

$$= \left[6\frac{y^6}{6} - 16\frac{y^4}{4} + 4\frac{y^2}{2} - 6\frac{y^4}{4}\right]_1^0$$

$$= 0 - \left(1 - 4 + 2 - \frac{3}{2}\right)$$

$$= -1 + 4 - 2 + \frac{3}{2}$$

$$= 1 + \frac{3}{2} = \frac{5}{2}.$$

$$\therefore L.H.S = \int_{OA} + \int_{AO}$$

$$= -1 + \frac{5}{2}$$

$$= \frac{-2+5}{2}$$

$$= \frac{3}{2}.$$

Hence Green's theorem is verified.

HW Verify Green's theorem in the xy plane for $\int_C \{(xy + y^2)\,dx + x^2 dy\}$, where C is the closed curve of the region bounded by $y = x$ and $y = x^2$ $\left(\frac{19}{20}, -1\right)$