Triple Integration in Cartesian Co-ordinates

Triple Integration of a function defined over a region $\iiint_R f(x,y,z)\,dx\,dy\,dz$.

1) $\int_0^3\int_0^2\int_0^1 (x+y+z)\,dz\,dy\,dx$.

$= \int_0^3\int_0^2 \left[\int_0^1 (x+y+z)\,dz\right] dy\,dx$.

$= \int_0^3\int_0^2 \left[xz+yz+\frac{z^2}{2}\right]_{z=0}^{z=1} dy\,dx$.

$= \int_0^3\int_0^2 \left[x+y+\frac{1}{2}\right] dy\,dx$.

$= \int_0^3 \left[xy+\frac{y^2}{2}+\frac{1}{2}y\right]_0^2 dx$

$= \int_0^3 \left(2x+\frac{4}{2}+\frac{2}{2}\right)-(0)\,dx$.

$= \int_0^3 \left(2x+\frac{4}{2}+1\right) dx$

$= \left[\frac{2x^2}{2}+2x+x\right]_0^3$

$= \left[3^2+2(3)+3\right] = 9+6+3 = 18$

$\boxed{\therefore I = 18}$

2) Evaluate $\int_0^1\int_0^2\int_0^3 xyz\,dx\,dy\,dz$

Soln:

$$I = \int_0^1\int_0^2\int_0^3 xyz\,dx\,dy\,dz.$$

$$= \int_0^1\int_0^2 \left[\frac{x^2}{2}\right]_0^3 yz\,dy\,dz$$

$$= \int_0^1\int_0^2 \left[\frac{3^2}{2}\right] yz\,dy\,dz.$$

$$= \frac{9}{2}\int_0^1 \left[\frac{y^2}{2}\right]_0^2 z\,dz.$$

$$= \frac{9}{2}\left[\frac{4}{2}\right]\int_0^1 z\,dz$$

$$= \frac{36}{4}\left[\frac{z^2}{2}\right]_0^1 = 9\left[\frac{1}{2}\right] = \frac{9}{2}$$

$$\therefore \boxed{I = \frac{9}{2}}$$

3) Evaluate $\int_0^a\int_0^b\int_0^c e^{x+y+z}\,dx\,dy\,dz$

Soln:

$$I = \int_0^a\int_0^b\int_0^c e^{x+y+z}\,dx\,dy\,dz.$$

$$= \int_0^a\int_0^b\int_0^c e^y e^z e^x\,dx\,dy\,dz$$

$$= \int_0^a\int_0^b e^y e^z \left[e^x\right]_0^c dy\,dz.$$

$$= \int_0^a\int_0^b e^y e^z \left[e^c - e^0\right) dy\,dz.$$

$$= (e^c-1)\int_0^a e^z\left(\int_0^b e^y\,dy\right) dz$$

$$= (e^c-1)\int_0^a e^z\left[e^y\right]_0^b dz$$

$$= (e^c-1)\int_0^a e^z(e^b-e^0)\,dz$$

$$= (e^c-1)\int_0^a e^z(e^b-1)\,dz$$

$$= (e^c-1)(e^b-1)\int_0^a e^z\,dz$$

$$= (e^c-1)(e^b-1)\left[e^z\right]_0^a$$

$$= (e^c-1)(e^b-1)(e^a-e^0)$$

$$= (e^c-1)(e^b-1)(e^a-1).$$

$$\therefore I = (e^c-1)(e^b-1)(e^a-1)$$