4) Using double Integration find the area between the curves $y^2 = 4x$ and $x^2 = 4y$

Soln:

$$\text{Area} = \iint_R dy\,dx \text{ or } \iint_R dx\,dy$$

Given curves: $\underbrace{y^2 = 4x}_{①}$; $\underbrace{x^2 = 4y}_{②}$

$② \Rightarrow \frac{x^2}{4} = y$

sub in ①.

$$\left(\frac{x^2}{4}\right)^2 = 4x$$

$$\frac{x^4}{16} = 4x$$

$$x^3 = 64$$

$$\boxed{x = 4}$$

$x = 4$ sub in ①

$$\Rightarrow y^2 = 4(4) = 16$$

$$\boxed{y = 4}$$

x limit: $x = 0$ to $x = 4$

y limit: $y = \frac{x^2}{4}$ to $y = 2\sqrt{x}$.

$$I = \int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} dy\,dx$$

$$= \int_0^4 [y]_{\frac{x^2}{4}}^{2\sqrt{x}}\,dx$$

$$= \int_0^4 \left[2\sqrt{x} - \frac{x^2}{4}\right] dx$$

$$= \left[2\frac{x^{3/2}}{3/2} - \frac{x^3}{4\times 3}\right]_0^4$$

$$= \left[2\times\frac{2}{3}x^{3/2} - \frac{1}{12}x^3\right]_0^4$$

$$= \left[\frac{4}{3}x^{3/2} - \frac{1}{12}x^3\right]_0^4$$

$$= \left[\frac{4}{3}(4)^{3/2} - \frac{1}{12}(4)^3\right] - [0]$$

$$= \frac{4}{3}(2^2)^{3/2} - \frac{1}{12}(4^3)$$

$$= \frac{4}{3}2^3 - \frac{64}{2} = \frac{32}{3} - \frac{64}{12}$$

$$= \frac{128 - 64}{12} = \frac{64}{12} = \frac{16}{3}.$$

$$\boxed{I = \frac{16}{3}}$$

5) Find the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Soln:

Given: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\text{Area} = \iint_R dy\,dx.$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 ; \quad \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}.$$

$$\frac{y^2}{b^2} = \frac{a^2 - x^2}{a^2}$$

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

$$y = \pm\frac{b}{a}\sqrt{a^2 - x^2}.$$

ommitting the negative value, since it is in the 1st quadrant

$$\therefore y = \frac{b}{a}\sqrt{a^2 - x^2}.$$

limits:

x limit ; $x = 0$ to $x = a$

y limit ; $y = 0$ to $y = \frac{b}{a}\sqrt{a^2 - x^2}$

Area of the ellipse

$= 4 \times$ Area of the 1^{st} quadrant

$$= 4 \times \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} dy\, dx = 4\int_0^a [y]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= 4 \times \int_0^a \left[\frac{b}{a}\sqrt{a^2-x^2}\right] - [0]\, dx$$

$$= 4 \times \int_0^a \frac{b}{a}\sqrt{a^2-x^2}\, dx.$$

$$= 4\frac{b}{a}\int_0^a \sqrt{a^2-x^2}\, dx$$

formula:

$$\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c.$$

$$\therefore I = 4\frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= 4\frac{b}{a}\left[\frac{a}{2}\sqrt{a^2-a^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{a}{a}\right)\right] - \left[0 + \frac{a^2}{2}\sin^{-1}(0)\right]$$

$$= \frac{4b}{a}\left[\frac{a}{2}(0) + \frac{a^2}{2}\sin^{-1}\left[\sin\left(\frac{\pi}{2}\right)\right]\right] - (0+0)$$

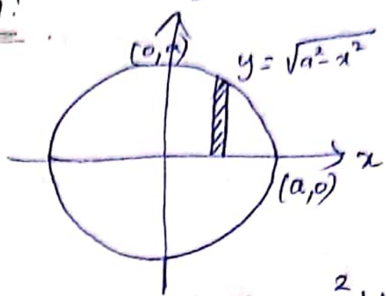
$$= \frac{\cancel{4}b}{\cancel{a}}\left[\frac{a^{\cancel{2}}}{\cancel{2}} \times \frac{\pi}{\cancel{2}}\right]$$

$$= \pi ab \,//$$

$\therefore$ Area of the ellipse $= \pi ab$.

5) Find the area of the circle of radius 'a' by double integration.

Soln:



We know that, $x^2+y^2=a^2$.

Area $= \iint dy\, dx$.

$$x^2+y^2=a^2$$
$$y^2=a^2-x^2$$
$$y=\pm\sqrt{a^2-x^2}$$

Considering only the 1^{st} quadrant, we have

$$y=\sqrt{a^2-x^2}$$

limits:

x limits: $x=0$ to $x=a$

y limits: $y=0$ to $y=\sqrt{a^2-x^2}$

Area of the circle.

$= 4 \times$ Area of 1^{st} quadrant

$$= 4 \times \int_0^a \int_0^{\sqrt{a^2-x^2}} dy\, dx$$

$$= 4\int_0^a [y]_0^{\sqrt{a^2-x^2}} dx$$

$$= 4\int_0^a \left(\sqrt{a^2-x^2} - 0\right) dx$$

$$= 4\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= 4\left[\frac{a}{2}\sqrt{a^2-a^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{a}{a}\right)\right] - 0$$

$$= 4\left[0+\frac{a^2}{2}\,\frac{\pi}{2}\right]$$

$$= 4 \times \frac{a^2}{2} \times \frac{\pi}{2} = \pi a^2 //$$

$\therefore$ Area of the circle $= \pi a^2 //$