(14) Find the values of a and b so that the surface $ax^3 - by^2z = (a+3)x^2$ and $4x^2y - z^3 = 11$ may cut orthogonally at $(2,-1,-3)$.

Soln:

$$a = -7/3 \quad \& \quad b = 64/9$$

DIVERGENCE OF A VECTOR POINT FUNCTION:

Let $\vec{F}$ be any given continuously differentiable vector point function then the divergence of $\vec{F}$ is defined as,

$$\text{div}\,\vec{F} = \nabla\cdot\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\vec{F}$$

$$= \vec{i}\cdot\frac{\partial\vec{F}}{\partial x} + \vec{j}\cdot\frac{\partial\vec{F}}{\partial y} + \vec{k}\cdot\frac{\partial\vec{F}}{\partial z}$$

Note :

1. $\nabla\cdot\vec{F}$ is a scalar point function.
2. If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ be a continuously differentiable vector point function then,

$$\text{div}\,\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Solenoidal Vector :

A vector $\vec{F}$ is said to be solenoidal vector if $\text{div}\,\vec{F} = 0$.

CURL OF A VECTOR POINT FUNCTION:

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ be any given continuously differentiable vector point function, the curl or rotation of $\vec{F}$ is defined as,

$$\text{Curl } \vec{F} = \nabla \times \vec{F}$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \vec{F}$$

$$= \vec{i} \times \frac{\partial \vec{F}}{\partial x} + \vec{j} \times \frac{\partial \vec{F}}{\partial y} + \vec{k} \times \frac{\partial \vec{F}}{\partial z}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Note : $\nabla \times \vec{F}$ is a vector point function.

IRROTATIONAL VECTOR :

A vector $\vec{F}$ is said to be irrotational if $\nabla \times \vec{F} = 0$

i.e., $\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = 0$

CONSERVATIVE VECTOR FIELD :

If a vector point function $\vec{F}$ is expressible as the gradient of a scalar point function φ, then $\vec{F}$ is conservative i.e., $\vec{F}$ is conservative if $\vec{F} = \nabla\varphi$. Here φ is called scalar potential.

$\vec{F}$ is conservative force if $\text{curl } \vec{F} = 0$.

PROBLEMS :

① Prove that $\text{curl}(\nabla\varphi) = 0$ (or) $\nabla \times \nabla\varphi = 0$.

Soln:

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$\text{Curl}(\nabla\varphi) = \nabla \times \nabla\varphi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z} \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial^2\varphi}{\partial y\,\partial z} - \frac{\partial^2\varphi}{\partial y\,\partial z}\right) - \vec{j}\left(\frac{\partial^2\varphi}{\partial x\,\partial z} - \frac{\partial^2\varphi}{\partial x\,\partial z}\right) + \vec{k}\left(\frac{\partial^2\varphi}{\partial x\,\partial y} - \frac{\partial^2\varphi}{\partial x\,\partial y}\right)$$

$$= 0$$

② Prove that $\text{div}(\text{curl}\,\vec{F}) = 0$ (or) $\nabla\cdot(\nabla\times\vec{F}) = 0$

Soln:

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$ if $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

$$\nabla\times\vec{F} = \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$\nabla\cdot\nabla\times\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left[\vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right]$$

$$\nabla \cdot \nabla \times \vec{F} = \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$= 0$$

(3) Show that curl grad $f = 0$ (or) $\nabla \times \nabla f = 0$.

Soln:

$$\text{Curl grad } f = \nabla \times \nabla f$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial^2 f}{\partial y\, \partial z} - \frac{\partial^2 f}{\partial y\, \partial z}\right] - \vec{j}\left[\frac{\partial^2 f}{\partial x\, \partial z} - \frac{\partial^2 f}{\partial x\, \partial z}\right] + \vec{k}\left[\frac{\partial^2 f}{\partial x\, \partial y} - \frac{\partial^2 f}{\partial x\, \partial y}\right]$$

$$= 0$$

(4) If $\nabla v = y\vec{i} + z\vec{j} + x\vec{k}$ what is the directional derivative of v at the point $(1, 2, 3)$ in the direction $3\vec{i} + 4\vec{j} + 5\vec{k}$.

Soln:

$$\nabla v = y\vec{i} + z\vec{j} + x\vec{k}$$

$$\nabla v_{(1,2,3)} = 2\vec{i} + 3\vec{j} + \vec{k}$$

$$\vec{a} = 3\vec{i} + 4\vec{j} + 5\vec{k}$$

$$|\vec{a}| = \sqrt{9 + 16 + 25} = \sqrt{50}$$

$$\text{Directional Derivative} = \nabla v \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$= (2\vec{i} + 3\vec{j} + \vec{k}) \cdot \frac{(3\vec{i} + 4\vec{j} + 5\vec{k})}{\sqrt{50}}$$

$$= \frac{6 + 12 + 5}{\sqrt{50}} = \frac{23}{\sqrt{50}}$$

(5) Prove that div $(\vec{u} \times \vec{v}) = \vec{v} \cdot \text{curl}\, \vec{u} - \vec{u} \cdot \text{curl}\, \vec{v}$

Soln:

$$\text{div}(\vec{u} \times \vec{v}) = \sum \vec{i}\, \frac{\partial}{\partial x} (\vec{u} \times \vec{v})$$

$$= \sum \vec{i} \cdot \left[\vec{u} \times \frac{\partial \vec{v}}{\partial x} + \frac{\partial \vec{u}}{\partial x} \times \vec{v} \right]$$

$$= \sum \vec{i} \cdot \left(\vec{u} \times \frac{\partial \vec{v}}{\partial x} \right) + \sum \vec{i} \cdot \left(\frac{\partial \vec{u}}{\partial x} \times \vec{v} \right)$$

$$= \left(\sum \vec{i} \times \frac{\partial \vec{u}}{\partial x} \right) \cdot \vec{v} - \left(\sum \vec{i} \times \frac{\partial \vec{v}}{\partial x} \right) \cdot \vec{u}$$

$$= \text{curl}\, \vec{u} \cdot \vec{v} - \text{curl}\, \vec{v} \cdot \vec{u}$$

$$= \vec{v}\, \text{curl}\, \vec{u} - \vec{u}\, \text{curl}\, \vec{v}$$

(6) Prove that $r^n \vec{r}$ is solenoidal if $n = -3$ and $r^n \vec{r}$ is irrotational for all values of n

Soln:

$$r^n \vec{r} = r^n (x\vec{i} + y\vec{j} + z\vec{k})$$

$$= r^n x \vec{i} + r^n y \vec{j} + r^n z \vec{k}$$

$$\text{div}(r^n \vec{r}) = \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$$

Now $r^2 = x^2 + y^2 + z^2$

$$2r \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = x/r$$

$$2r \frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = y/r$$

$$2r \frac{\partial r}{\partial z} = 2z \Rightarrow \frac{\partial r}{\partial z} = z/r$$

Now $\frac{\partial}{\partial x}(r^n x) = x \frac{\partial}{\partial r}(r^n) \frac{\partial r}{\partial x} + r^n$

$$= x n r^{n-1} \frac{x}{r} + r^n = x^2 n r^{n-2} + r^n$$

$$\frac{\partial}{\partial y}(r^n y) = y^2 n r^{n-2} + r^n$$

$$\frac{\partial}{\partial z}(r^n z) = z^2 n r^{n-2} + r^n$$

$$\text{div}(r^n \vec{r}) = (x^2+y^2+z^2) n r^{n-2} + 3r^n$$

$$= r^2 n r^{n-2} + 3r^n$$

$$= n r^n + 3r^n$$

$$= (n+3) r^n$$

The vector $r^n \vec{r}$ is solenoidal if,

$$\text{div}(r^n \vec{r}) = 0$$

$$\Rightarrow (n+3) r^n = 0$$

$$\Rightarrow n+3 = 0 \Rightarrow \boxed{n = -3}$$

$\therefore r^n \vec{r}$ is solenoidal only if $n = -3$.

$$\text{curl}(r^n \vec{r}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= \Sigma \vec{i} \frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n y)$$

$$= \Sigma \vec{i} \left\{ n r^{n-1} \frac{\partial r}{\partial y} \cdot z - n r^{n-1} \frac{\partial r}{\partial z} \cdot y \right\}$$

$$= \Sigma \vec{i} \left\{ n r^{n-1} \left(\frac{yz}{r} - \frac{z}{r} y \right) \right\}$$

$$= 0$$

$\therefore \text{curl}(r^n \vec{r}) = \vec{0}$ for all values of n.

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(7) Find the constants a, b, c so that the vector

$\vec{F} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$

is irrotational.

Soln:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(4x+cy+2z) - \frac{\partial}{\partial z}(bx-3y-z)\right]$$

$$-\vec{j}\left[\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}(bx-3y-z) - \frac{\partial}{\partial y}(x+2y+az)\right]$$

$$= \vec{i}(c+1) - \vec{j}(4-a) + \vec{k}(b-2).$$

Given : $\vec{F}$ is irrotational.

i.e., $\nabla \times \vec{F} = 0$.

$\vec{i}(c+1) - \vec{j}(4-a) + \vec{k}(b-2) = 0$.

$c+1 = 0 \Rightarrow c = -1$

$4-a = 0 \Rightarrow a = 4$

$b-2 = 0 \Rightarrow b = 2$

(8) Find 'a' so that the vector

$\vec{A} = (ax^2 - y^2 + x)\vec{i} - (2xy+y)\vec{j}$ is irrotational.

Soln:

Given : $\vec{A}$ is irrotational.

$\nabla \times \vec{A} = 0$.

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ax^2 - y^2 + x & -(2xy+y) & 0 \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(-2y + 2y) = 0.$$

$\therefore$ 'a' is arbitrary.

⑨ prove $\vec{F} = (y^2 \cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} + 3xz^2\vec{k}$ is irrotational and find its scalar potential φ such that $\vec{F} = \nabla\varphi$.

Soln:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2\cos x + z^3 & 2y\sin x - 4 & 3xz^2 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(3xz^2) - \frac{\partial}{\partial z}(2y\sin x - 4)\right]$$

$$- \vec{j}\left[\frac{\partial}{\partial x}(3xz^2) - \frac{\partial}{\partial z}(y^2\cos x + z^3)\right]$$

$$+ \vec{k}\left[\frac{\partial}{\partial x}(2y\sin x - 4) - \frac{\partial}{\partial y}(y^2\cos x + z^3)\right]$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k}$$

$$= 0$$

$\therefore \vec{F}$ is irrotational.

To find φ:

$$\nabla\varphi = (y^2\cos x + z^2)\vec{i} + (2y\sin x - 4)\vec{j} + 3xz^2\vec{k}$$

We know that $\nabla\varphi = \vec{i}\dfrac{\partial\varphi}{\partial x} + \vec{j}\dfrac{\partial\varphi}{\partial y} + \vec{k}\dfrac{\partial\varphi}{\partial z}$

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$$\frac{\partial \varphi}{\partial x} = y^2 \cos x + z^3 \Rightarrow \varphi = y^2 \sin x + z^3 x + f(y,z)$$

$$\frac{\partial \varphi}{\partial y} = 2y \sin x - 4 \Rightarrow \varphi = y^2 \sin x - 4y + f(x,z)$$

$$\frac{\partial \varphi}{\partial z} = 3xz^2 \Rightarrow \varphi = xz^3 + f(x,y)$$

(10) Show that $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$ is irrotational. Find φ such that $\vec{F} = \nabla \varphi$.

Soln: $\varphi = 3x^2 y + xz^3 - yz + c$

(11) If $\nabla \varphi = yz\vec{i} + xz\vec{j} + xy\vec{k}$ then find φ.

Soln: $\varphi = xyz + c$

(12) Prove that $\operatorname{div} \hat{r} = 2/r$.

Soln:

$$\operatorname{div} \hat{r} = \nabla \cdot \left(\frac{\vec{r}}{r}\right)$$

$$= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot \left(\frac{x\vec{i} + y\vec{j} + z\vec{k}}{r}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r}\right)$$

$$= \frac{1}{r} - \frac{1}{r^2} \cdot x \frac{\partial r}{\partial x} + \frac{1}{r} - \frac{1}{r^2} y \frac{\partial r}{\partial y} + \frac{1}{r} - \frac{1}{r^2} z \frac{\partial r}{\partial z}$$

$$= \frac{3}{r} - \frac{1}{r^2}\left[x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right]$$

Now $r^2 = x^2 + y^2 + z^2$

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$2r\frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$2r\frac{\partial r}{\partial z} = 2z \Rightarrow \frac{\partial r}{\partial z} = \frac{z}{r}$$

Now $\text{div } \hat{r} = \frac{3}{r} - \frac{1}{r^2}\left[x\cdot\frac{x}{r} + y\cdot\frac{y}{r} + z\cdot\frac{z}{r}\right]$

$$= \frac{3}{r} - \frac{1}{r^2}\left[\frac{x^2+y^2+z^2}{r}\right]$$

$$= \frac{3}{r} - \frac{1}{r^2}\cdot\frac{r^2}{r} = \frac{3}{r} - \frac{1}{r}$$

$$= \frac{2}{r}$$

(13) Prove that $(\text{curl curl } \vec{F}) = \nabla(\text{div } \vec{F}) - \nabla^2\vec{F}$.

Soln:

Given: $\nabla\times(\nabla\times\vec{F}) = (\nabla\cdot\vec{F})\nabla - (\nabla\cdot\nabla)\vec{F}$

$$\left[\because \vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}\right]$$

$$= \nabla(\nabla\cdot\vec{F}) - \nabla^2\vec{F}$$

$$= \nabla(\text{div}\vec{F}) - \nabla^2\vec{F}.$$

(14) Prove that $\vec{F} = (2xy+z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$ is a conservative force and hence find φ so that $\vec{F} = \nabla\varphi$.

Soln:

$\vec{F}$ is a conservative force i.e., $\text{Curl } \vec{F} = 0$.

& $\varphi = x^2y + xz^3 + c$ where c is any arbitrary constant.