Area as the double integral

Formula :

* If $\iint dx\, dy$ is given then, draw the strip parallel

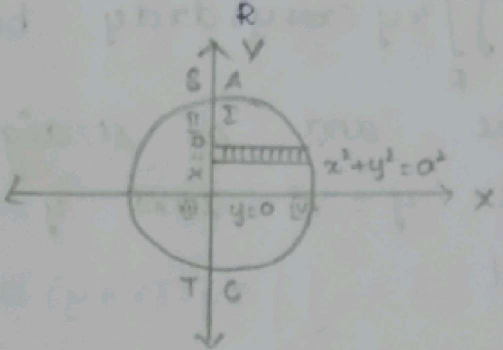
to x - axis

* If $\iint dy\,dx$ is given then, draw a strip parallel to y-axis

Example:

Evalute $\iint xy \, dx\, dy$ where R is the First Quadrant of the circle $x^2+y^2=a^2$ $(x>=0,\ y>=0)$

Soln: Let $I = \iint xy\, dx\, dy$ +ve qud $x^2+y^2=a^2$



$x^2+y^2=a^2$
$x^2=a^2-y^2$
$x=\sqrt{a^2-y^2}$

* x varies from $x=0$ to $x=\sqrt{a^2-y^2}$
* y varies from $y=0$ to $y=a$

$$I = \int_0^a \int_0^{\sqrt{a^2-y^2}} xy \, dx\, dy$$

$$= \int_0^a \left[y \int_0^{\sqrt{a^2-y^2}} x\, dx \right] dy$$

$$= \int_0^a y \left[\frac{x^2}{2}\right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \int_0^a y \left[\frac{a^2-y^2}{2}\right] dy$$

$$= \frac{1}{2}\left[\int_0^a y(a^2-y^2)\, dy\right]$$

$$= \frac{1}{2}\left[a^2\int_0^a y\, dy - \int_0^a y^3 dy\right]$$

$$= \frac{1}{2}\left[a^2\left[\frac{y^2}{2}\right]_0^a - \left[\frac{y^4}{4}\right]_0^a\right]$$

$$= \frac{1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right]$$

$$= \frac{1}{2}\left[\frac{4a^4 - 2a^4}{8}\right]$$

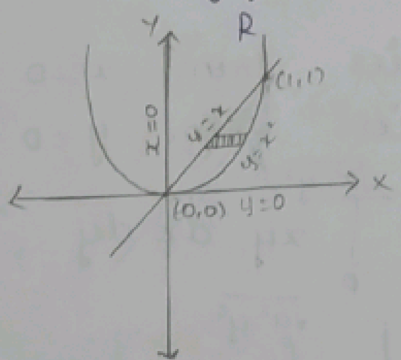
$$= \frac{1}{2}\left[\frac{2a^4}{8}\right]$$

$$= \frac{a^4}{8}$$

Example : 2

Evalute $\iint_R xy\,(x+y)\,dx\,dy$ between the curve and a straight line such as $y = x^2$ and $y = x$

Soln: Let $I = \iint_R xy\,(x+y)\,dx\,dy$



$x = y \rightarrow (1)$

$x^2 = y \rightarrow (2)$

Take (2) $y = x$

sub (1) in (2)

$x^2 = x$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0$ (or) $x - 1 = 0 \Rightarrow x = 1$

sub $x = 0$ in (1)

$y = 0$

sub $x = 1$ in (2)

$y = 1$

* x varies from $x = y$ to $x = \sqrt{y}$

* y varies from $y = 0$ to $y = 1$

$$I = \int_0^1 \int_y^{\sqrt{y}} (x^2 y + x y^2)\,dx\,dy$$

$$= \int_0^1 \left[y \int_y^{\sqrt{y}} x^2\, dx + y^2 \int_y^{\sqrt{y}} x\, dx \right] dy$$

$$= \int_0^1 \left[y \left[\frac{x^3}{3}\right]_y^{\sqrt{y}} + y^2 \left[\frac{x^2}{2}\right]_y^{\sqrt{y}} \right] dy$$

$$= \int_0^1 \frac{1}{3}\left[y^2\sqrt{y} - y^4 \right] + \frac{1}{2}\left[y^3 - y^4 \right] dy$$

$\cancel{45} \times 3$ $\cancel{225}$ $\cancel{4200}$ $\cancel{840}$ $\cancel{168}$ 56

$$= \frac{1}{3}\int_0^1 y^{5/2}\, dy - \frac{1}{3}\int_0^1 y^4\, dy + \frac{1}{2}\int_0^1 y^3\, dy - \frac{1}{2}\int_0^1 y^4\, dy$$

$$= \frac{1}{3}\left[\frac{y^{7/2}}{7/2}\right]_0^1 - \frac{1}{3}\left[\frac{y^5}{5}\right]_0^1 + \frac{1}{2}\left[\frac{y^4}{4}\right]_0^1 - \frac{1}{2}\left[\frac{y^5}{5}\right]_0^1$$

$$= \frac{1}{3}\left(\frac{2}{7}\right) - \frac{1}{3}\left(\frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{4}\right) - \frac{1}{2}\left[\frac{1}{5}\right]$$

$$= \frac{2}{21} - \frac{1}{15} + \frac{1}{8} - \frac{1}{10}$$ $\frac{\cancel{37}^{1}}{168}$

$$= \frac{2}{21} + \frac{1}{8} - \left[\frac{1}{15} + \frac{1}{10}\right]$$

$$= \frac{16 + 21}{168} - \left[\frac{10 + 15}{150}\right]$$

$$= \frac{37}{168} - \frac{25}{150}$$

$$= \frac{3}{56}.$$

Example : 3

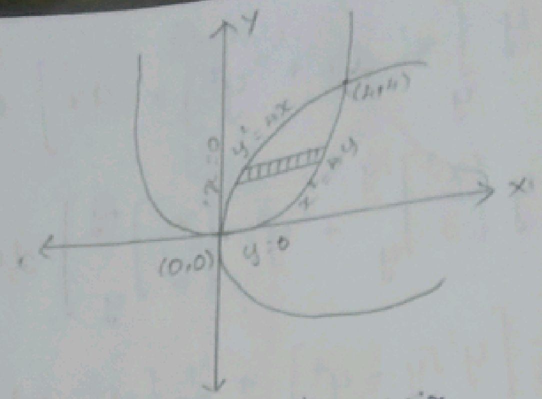
Find the area between the $\iint xy^2 = 4x$ and $x^2 = 4y$

If they don't given function we can take $\iint dx\,dy$ (or) $\iint dy\,dx$ [both have same ans]

Soln: $y^2 = 4x$ and $x^2 = 4y$

$\sqrt{y}$

Let $I = \iint_R dx\, dy$



$$y^2 = 4x \to (1)$$
$$x^2 = 4y \to (2)$$

Take (2) $x^2 = 4y$

$$\frac{x^2}{4} = y$$

sub y in (1)

$$y^2 = 4x$$
$$\left[\frac{x^2}{4}\right]^2 = 4x$$
$$\frac{x^4}{16} = 4x$$

$$x^4 = 64x$$
$$x^3 = 4^3$$
$$x = 4$$

sub $x = 4$ in (2)

$$x^2 = 4y$$
$$4^2 = 4y$$
$$4y = 16$$
$$y = 4$$

* x varies from $x = \frac{y^2}{4}$ to $x = 2\sqrt{y}$
* y varies from $y = 0$ to $y = 4$

$$I = \int_0^4 \int_{y^2/4}^{2\sqrt{y}} dx\, dy$$
$$= \int_0^4 \left[\int_{y^2/4}^{2\sqrt{y}} dx\right] dy$$
$$= \int_0^4 (x]_{y^2/4}^{2\sqrt{y}}\, dy$$
$$= \int_0^4 (2\sqrt{y} - \frac{y^2}{4}]\, dy$$
$$= 2\int_0^4 \sqrt{y}\, dy - \frac{1}{4}\int_0^4 y^2\, dy$$
$$= 2\left[\frac{y^{3/2}}{3/2}\right]_0^4 - \frac{1}{4}\left[\frac{y^3}{3}\right]_0^4$$
$$= 2\left[\frac{2 \times 4^{3/2}}{3}\right] - \frac{1}{4}\left[\frac{4^3}{3}\right]$$

$$= \left[\frac{4\times 8}{3} - \frac{64}{12}\right]$$

$$= \frac{32}{3} - \frac{64}{12}$$

$$= \frac{384 - 192}{36}$$

$$= \frac{192}{36}$$

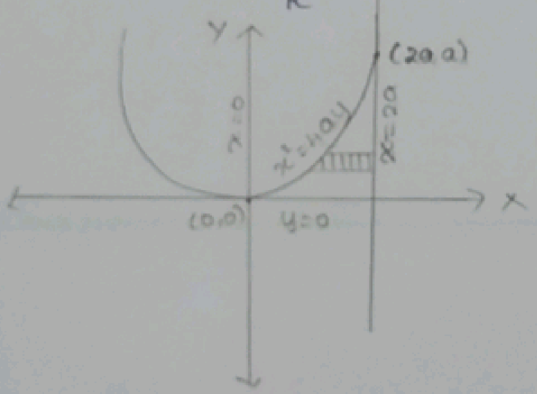
$$= \frac{16}{3}$$

Example: 4

Evalute $\iint_R xy \, dx\, dy$ where A is the region bounded by $x = 2a$ $x^2 = 4ay$

Soln: $x = 2a$ and $x^2 = 4ay$

Let $I = \iint_R xy \, dx \, dy$



$x^2 = 4ay \rightarrow (1)$

$x = 2a \longrightarrow (2)$

sub (2) in (1)

$(2a)^2 = 4ay$

$4a^2 = 4ay$

$y = a$

sub $y = a$ in (1)

$x^2 = 4a(a)$

$x^2 = 4a^2$

$x = 2a$.

point $(2a, a)$

* x varies from $2\sqrt{ay}$ to $2a$

ie, $x = 2\sqrt{ay}$ to $x = 2a$

* y varies from 0 to a

ie, $y = 0$ to $y = a$

$$I = \int_{0}^{a}\int_{2\sqrt{ay}}^{2a} xy \, dx \, dy$$

$$= \int_{0}^{a} \left[\int_{2\sqrt{ay}}^{2a} xy \, dx \right] dy$$

$$= \int_{0}^{a} \left[\left[\frac{x^2 y}{2} \right]_{2\sqrt{ay}}^{2a} \right] dy$$

$$= \int_{0}^{a} \left[\frac{4a^2 y}{2} - \frac{4ay^2}{2} \right] dy$$

$$= \int_{0}^{a} \frac{4a^2 y}{2} \, dy - \int_{0}^{a} \frac{4ay^2}{2} \, dy$$

$$= \frac{4a^2}{2} \left[\frac{y^2}{2} \right]_{0}^{a} - \frac{4a}{2} \left[\frac{y^3}{3} \right]_{0}^{a}$$

$$\frac{24a^4 - 16a^4}{24}$$

$$\frac{8a^4}{24}$$

$$= \frac{4a^4}{4} - \frac{4a^4}{6}$$

$$= \frac{8a^4}{24}$$

$$= \frac{a^4}{3}$$