Triple integral: (volume)

Example: 1.

Evalute $\int_0^1 \int_0^1 \int_0^1 (x+y+z)\, dz\, dy\, dx$.

Soln: Let $I = \int_0^1 \int_0^1 \int_0^1 (x+y+z)\, dz\, dy\, dx$.

$$= \int_0^1 \int_0^1 \left[\int_0^1 x\, dz + \int_0^1 y\, dz + \int_0^1 z\, dz \right] dy\, dx$$

$$= \int_0^1 \int_0^1 \left[(xz)_0^1 + (yz)_0^1 + \left[\frac{z^2}{2}\right]_0^1 \right] dy\, dx$$

$$= \int_0^1 \left[\int_0^1 (x+y+\tfrac{1}{2}]\, dy \right] dx.$$

$$= \int_0^4 \left[\int_0^1 x\,dy + \int_0^1 y\,dy + \frac{1}{2}\int_0^1 dy \right] dx$$

$$= \int_0^4 \left[(xy)_0^1 + \left(\frac{y^2}{2}\right)_0^1 + \left(y/2\right)_0^1 \right] dx$$

$$= \int_0^4 (x + 1/2) + 1/2)\,dx$$

$$= \int_0^4 (x+1)\,dx$$

$$= \int_0^4 x\,dx + \int_0^4 dx$$

$$= \left[\frac{x^2}{2}\right]_0^4 + (x]_0^4$$

$$= \frac{16}{2} + 4$$

$$= 12.$$

Example : 2

$$\int_0^a \int_0^b \int_0^c x + y + z \; dz \; dy \; dx.$$

Soln: Let $I = \int_0^a \int_0^b \int_0^c x + y + z \; dz \, dy \, dx$

$$= \int_0^a \int_0^b \left[\int_0^c x \, dz + \int_0^c y \, dz + \int_0^c z \, dz \right] dy \, dx.$$

$$= \int_0^a \int_0^b \left[(xz)_0^c + (yz)_0^c + \left[\frac{z^2}{2}\right]_0^c \right] dy \, dx$$

$$= \int_0^a \int_0^b \left[cx + cy + \frac{c^2}{2} \right] dy \, dx.$$

$$= \int_0^a \left[\int_0^b cx \, dy + \int_0^b cy \, dy + \int_0^b \frac{c^2}{2} dy \right] dx.$$

$$= \int_0^a \left[(cxy)_0^b + \left[\frac{cy^2}{2}\right]_0^b + \left[\frac{c^2y}{2}\right]_0^b \right] dx$$

$$= \int_0^a \left[cbx + \frac{cb^2}{2} + \frac{c^2b}{2} \right] dx$$

$$= \int_0^a cbx \, dx + \int_0^a \frac{cb^2}{2} dx + \int_0^a \frac{c^2b}{2} dx$$

$$= \left[\frac{cbx^2}{2}\right]_0^a + \left[\frac{cb^2x}{2}\right]_0^a + \left[\frac{c^2bx}{2}\right]_0^a$$

$$= \frac{a^2bc}{2} + \frac{ab^2c}{2} + \frac{abc^2}{2}$$

$$= abc \left[\frac{a+b+c}{2}\right]$$

Example: 3

$$\int_0^1 \int_0^y \int_0^{x+y} dx \, dy \, dz$$

Soln:

Let $I = \int_0^1 \int_0^y \int_0^{x+y} dx \, dy \, dz$

$$= \int_0^1 \int_0^y \int_0^{x+y} dz \, dx \, dy$$

$$= \int_0^1 \int_0^y \left[\int_0^{x+y} dz \right] dx \, dy$$

$$= \int_0^1 \int_0^y \left[(z)_0^{x+y} \right] dx \, dy$$

$$= \int_0^1 \int_0^y (x+y) \, dx \, dy$$

$$= \int_0^1 \left[\int_0^y x\,dx + \int_0^y y\,dx \right] dy$$

$$= \int_0^1 \left(\left[\frac{x^2}{2}\right]_0^y + \cancel{\left[\frac{xy}{2}\right]} (xy)_0^y \right] dy$$

$$= \int_0^1 \left[\frac{y^2}{2} + y^2 \right] dy$$

$$= \int_0^1 \frac{y^2}{2}\,dy + \int_0^1 y^2\,dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \left[\frac{y^3}{3}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{3}\right] + \frac{1}{3}$$

$$= \frac{1}{6} + \frac{1}{3}$$

$$= \frac{3+6}{18}$$

$$= 9/18$$

$$= 1/2$$

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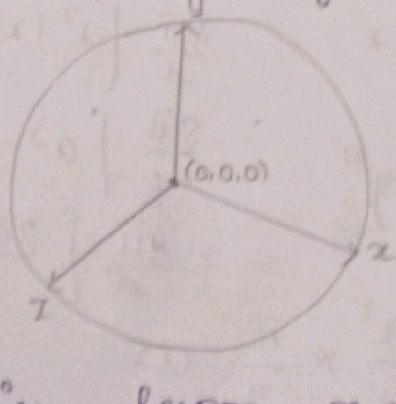
Example : 4

Find the volume of the shpere $x^2+y^2+z^2=a^2$ without transformation

Soln: Let $I \Rightarrow x^2+y^2+z = a^2$



$x^2+y^2+z^2=a^2$

$\boxed{z=\sqrt{a^2-x^2-y^2}}$

(Here z is vanished because we already taken]

$x^2+y^2=a^2$

$\boxed{y=\sqrt{a^2-x^2}}$

$x^2=a^2$

$\boxed{x=a}$

x varies from $x=0$ to $x=a$

y varies from $y=0$ to $y=\sqrt{a^2-x^2}$

z varies from $z=0$ to $z=\sqrt{a^2-x^2-y^2}$

Volume $= 8\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx$

$= 8\int_0^a\int_0^{\sqrt{a^2-x^2}} [z]_0^{\sqrt{a^2-x^2-y^2}}\,dy\,dx.$

$= 8\int_0^a\int_0^{\sqrt{a^2-x^2}} \sqrt{(\sqrt{a^2-x^2})^2-y^2}\,dy\,dx$

$\therefore \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left[\frac{x}{a}\right]$

Take $a^2 = (\sqrt{a^2-x^2})^2$, $x^2 = y^2$

$a = \sqrt{a^2-x^2}$, $x = y$

$= 8\int_0^a\left[\frac{y}{2}\sqrt{a^2-y^2}+\frac{(a^2-x^2)}{2}\sin^{-1}\frac{y}{\sqrt{a^2-x^2}}\right]_0^{\sqrt{a^2-x^2}}dx.$

$= 8\int_0^a\left[\frac{\sqrt{a^2-x^2}}{2}\sqrt{a^2-(\sqrt{a^2-x^2})^2}+\frac{a^2-x^2}{2}\sin^{-1}\frac{\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}}\right]dx$

$= 8\int_0^a\left[0+\frac{a^2-x^2}{2}\sin^{-1}(1)\right]dx$

$= 8\int_0^a\left[\frac{a^2}{2}\times\frac{\pi}{2}\right]dx$ $\quad 8\int_0^a\frac{a^2-x^2}{2}\times\frac{\pi}{2}\,dx.$

$= 8\int_0^a\frac{\pi a^3}{4}\,dx$ $\quad 8\times\frac{\pi}{2}\times\frac{1}{2}\int_0^a a^2-x^2\,dx$

$\frac{8\pi}{4}\left[\int_0^a a^2\,dx-\int_0^a x^2\,dx\right.$

$= \frac{8\pi a^3}{4}\int_0^a dx$ $\quad \frac{8\pi}{4}\left[a^2(x)_0^a-\left[\frac{x^3}{3}\right]_0^a\right)$

$\frac{8\pi}{4}\left[a^3-\frac{a^3}{3}\right]$

$= \frac{8\pi a^3}{4}[x]_0^a$ $\quad \frac{8\pi}{4}\left[\frac{3a^3-a^3}{3}\right] = \frac{8\pi}{2}\frac{2a^3}{3}$

$\frac{4\pi a^3}{3}$

$= 8\int_0^a\frac{a^2-x^2}{2}\times\frac{\pi}{2}\,dx.$

$= 8\times\frac{\pi}{4}\left[\int_0^a a^2\,dx-\int_0^a x^2\,dx\right]$

$$= 8\pi \left[(a^2 x)_0^a - \left[\frac{x^3}{3}\right]_0^a \right]$$

$$= 8\pi \left[a^3 - \frac{a^3}{3}\right]$$

$$= 8\pi \left[\frac{3a^3 - a^3}{3}\right]$$

$$= \frac{4\pi a^3}{3}$$

Example : 5

Evalute $\iiint_V dx\, dy\, dz$ where V is the region of space inside the Tetrahedron bounded by the planes $x=0,\ y=0,\ z=0$ and $x/a + y/b + z/c = 1$

Soln: Given,

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

x varies from $x=0$ to $x=a$

y varies from $y=0$ to $y = b\left[1 - \frac{x}{a}\right]$

z varies from $z=0$ to $z = c\left[1 - \frac{x}{a} - \frac{y}{b}\right]$

$$\text{Volume} = \int_0^a \int_0^{b[1-x/a]} \int_0^{c[1-x/a-y/b]} dz\, dy\, dx.$$

$$= \int_0^a \int_0^{b[1-x/a]} [z]_0^{c[1-x/a-y/b]} dy\, dx.$$

$$= \int_0^a \int_0^{b[1-x/a]} c\left[1 - \frac{x}{a} - \frac{y}{b}\right] dy\, dx.$$

$$= \int_0^a c\left[\int_0^{b[1-x/a]} 1\, dy - \frac{x}{a}\int_0^{b[1-x/a]} dy - \frac{1}{b}\int_0^{b[1-x/a]} y\, dy\right] dx$$

$$= \int_0^a c\left[[y]_0^{b[1-x/a]} - \left(\frac{xy}{a}\right)_0^{b[1-\frac{x}{a}]} - \left[\frac{y^2}{2b}\right]_0^{b[1-\frac{x}{a}]} \right] dx$$

$$= \int_0^a \left[c\left[\left[b\left(1-\frac{x}{a}\right)\right] - \frac{x\left[b\left(1-x/a\right)\right]}{a} - \frac{b^2\left(1-x/a\right)^2}{2b}\right] dx\right.$$

$$= c\int_0^a b\left[(1-x/a) - \frac{x(1-x/a)}{a} - \frac{b(1-x/a)^2}{2b}\right]$$

$$\cancel{= cb\int_0^a (1-x/a)\left[1-\frac{x}{a} - \frac{b(1-x/a)}{2b}\right] dx}$$

$$= cb\int_0^a \left[1-\frac{x}{a}-\frac{x}{a}+\frac{x^2}{a^2}+\frac{-1-\frac{x^2}{a^2}+2\left(\frac{x}{a}\right)}{2}\right]dx$$

$$= cb\int_0^a \left[1-\frac{x}{a}-\frac{x}{a}+\frac{x^2}{a^2}-\frac{1}{2}-\frac{x^2}{2a^2}+\frac{2x}{2a}\right]dx$$

$$= bc\int_0^a \left[\frac{1}{2}-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^2}{2a^2}\right]dx$$

$$= bc\int_0^a \frac{1}{2}\,dx - \int_0^a \frac{x}{a}\,dx + \int_0^a \frac{x^2}{a^2}\,dx - \int_0^a \frac{x^2}{2a^2}\,dx$$

$$= bc\left[\frac{1}{2}[x]_0^a - \frac{1}{a}\left[\frac{x^2}{2}\right]_0^a + \frac{1}{a^2}\left[\frac{x^3}{3}\right] - \frac{1}{2a^2}\left[\frac{x^3}{3}\right]_0^a\right.$$

$$= bc\left[\frac{a}{2}-\frac{a}{2}+\frac{a}{3}-\frac{a}{6}\right]$$

$$= bc\left[\frac{6a-3a}{18}\right]$$

$$= abc\left[\frac{3}{18}\right]$$

$$= \frac{abc}{6}$$