UNIT - IV

COMPLEX INTEGRATION

Cauchy's integral theorem (or) Cauchy's fundamental theorem :

If a function $f(z)$ is analytic and its derivative $f'(z)$ is continuous at all points inside and on a simple closed curve c, then

$$\boxed{\int_c f(z)\,dz = 0}$$

Cauchy's integral formula :

If $f(z)$ is analytic inside and on a simple closed curve 'c' and 'a' be any point inside c then,

$$\boxed{\int_c \frac{f(z)}{z-a}\,dz = 2\pi i \cdot f(a)}$$

Note : If the point lies outside c then

$$\int_c \frac{f(z)}{z-a}\,dz = 0 .$$

Cauchy's integral formula for derivatives :

If $f(z)$ is analytic inside and on a simple closed curve 'c' and 'a' be any point inside c then,

$$\oint_C \frac{f(z)}{(z-a)^2}\,dz = \frac{2\pi i}{1!} f'(a)$$

$$\int_C \frac{f(z)}{(z-a)^3}\,dz = \frac{2\pi i}{2!} f''(a)$$

In general,

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^n(a)$$

Problems :

1. Evaluate $\int_C \frac{\cos \pi z}{z-1}\,dz$ if c is $|z| = 2$

Soln :

Formula :

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i . f(a)$$

$z - a = 0$

$z - 1 = 0$

$z = 1$

Given : $|z| = 2$

$1 < 2$

$\therefore$ It lies inside c.

$$\therefore \int_C \frac{\cos \pi z}{z-1}\,dz = 2\pi i . f(1)$$

(Here $a = 1$

$f(z) = \cos \pi z$)

$= 2\pi i \cos \pi$

$= 2\pi i (-1)$

$= -2\pi i$

(2) Evaluate $\int_C \frac{z\,dz}{z-2}$ where C is the circle $|z| = 1$.

Soln:

$$f(z) = z/z-2$$

Here $z = 2$ lies outside C. $\left(\because \begin{array}{l} |z| = 1 \\ 2 > 1 \end{array} \right)$

$$\therefore \int_C f(z)\,dz = 0$$

$$\Rightarrow \int_C \frac{z\,dz}{z-2} = 0.$$

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(3) Evaluate $\int_C \frac{z}{(z-1)^3}\,dz$ where C is $|z| = 2$.

Soln:

$$z - 1 = 0$$

$$z = 1$$

Given: $|z| = 2 \Rightarrow |1| = 1 < 2$.

$\therefore$ It lies inside C.

Formula: $\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^n(a)$

$$\therefore \int_C \frac{f(z)}{(z-1)^3} = \frac{2\pi i}{2!} f''(1)$$

$$= \frac{\cancel{2}\pi i}{\cancel{2}} f''(1)$$

$$= \pi i\,(0)$$

$$= 0.$$

$f(z) = z$

$f'(z) = 1$

$f''(z) = 0$

$f''(1) = 0$.

④ Using Cauchy's integral formula evaluate

$$\int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz$$ where c is $|z| = \frac{3}{2}$

Soln: Using partial fractions,

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} \longrightarrow ①.$$

$$1 = A(z-2) + B(z-1)$$

Put $z = 1 \Rightarrow$ $1 = A(1-2) + 0.$

$1 = A(-1)$

$\boxed{A = -1}$

Put $z = 2 \Rightarrow 1 = A(0) + B(2-1)$

$\boxed{1 = B}$

$$\therefore ① \Rightarrow \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

Multiply by $\cos \pi z^2$ on both sides,

$$\frac{\cos \pi z^2}{(z-1)(z-2)} = \frac{-\cos \pi z^2}{z-1} + \frac{\cos \pi z^2}{z-2}$$

$$\therefore \int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz = \int_C \frac{-\cos \pi z^2}{z-1} dz + \int_C \frac{\cos \pi z^2}{z-2} dz$$

$\longrightarrow ②.$

$z = 1$; $z = 2$

$1 < 3/2$; $2 > 3/2$

$$\therefore \int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz = -2\pi i \, f(1) + 0$$

$$= -2\pi i \, (-1) \qquad \left[\begin{aligned} f(z) &= \cos \pi z^2 \\ f(1) &= \cos \pi \\ &= -1 \end{aligned} \right]$$

$$= 2\pi i$$

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(5) Evaluate $\displaystyle\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ where C is $|z| = 3$.