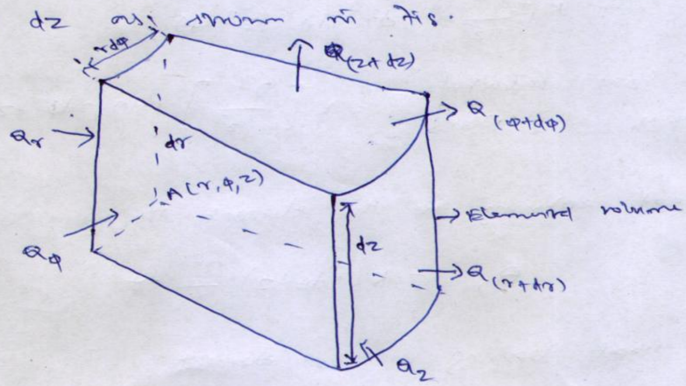
3

General heat conduction equation in cylindrical co-ordinates:-

consider a small cylindrical element of sides dr, $d\phi$ and dz as shown in fig.



$$\left\{\begin{matrix}\text{Net heat} \\ \text{conducted into element} \\ \text{from all the co-ordinate} \\ \text{directions}\end{matrix}\right\} + \left\{\begin{matrix}\text{heat} \\ \text{generated} \\ \text{within the} \\ \text{element}\end{matrix}\right\} = \left\{\begin{matrix}\text{heat stored} \\ \text{in the} \\ \text{element}\end{matrix}\right\} \quad \text{—(1)}$$

Net heat conducted into element from all co-ordinate directions:-

Heat entering in the element through (r,ϕ) plane in time $d\theta$.

$$Q_z = -k(r\,d\phi\,dr)\frac{\partial T}{\partial z}\,d\theta$$

Heat leaving from the element through (r,ϕ) plane in time $d\theta$

$$Q_{z+dz} = Q_z + \frac{\partial}{\partial z}(Q_z)\,dz$$

Net heat conducted

$$= Q_z - Q_{z+dz}$$
$$= -\frac{\partial}{\partial z}(Q_z)\,dz$$

$$= \frac{\partial}{\partial z}\left[k\,(r\,d\phi \cdot dr)\cdot\left[\frac{\partial T}{\partial z}\right]d\theta\right]dz$$

$$= k\left(\frac{\partial^2 T}{\partial z^2}\right)(dr \cdot r\,d\phi \cdot dz)\,d\theta$$

Net heat conducted through (r, ϕ) plane $= k\left[\frac{\partial^2 T}{\partial z^2}\right](dr \cdot r\,d\phi \cdot dz)\,d\theta$ — (2)

Heat entering in the element through (ϕ, z) plane in time $d\theta$

$$Q_r = -k\,(r\,d\phi\,dz)\,\frac{\partial T}{\partial r}\,d\theta$$

Heat leaving in the element

$$Q_{r+dr} = Q_r + \frac{\partial}{\partial r}(Q_r)\,dr$$

Net heat conducted

$$= Q_r - Q_{r+dr}$$

$$= -\frac{\partial}{\partial r}(Q_r)\,dr$$

$$= -\frac{\partial}{\partial r}\left[-k\,(r\,d\phi \cdot dz)\left(\frac{\partial T}{\partial r}\right)d\theta\right]dr$$

$$= k\,(dr\,d\phi\,dz)\cdot\frac{\partial}{\partial r}\left(r\cdot\frac{\partial T}{\partial r}\right)d\theta$$

$$= k\,(dr \cdot r\,d\phi \cdot dz)\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right]d\theta$$ — (3)

Heat entering in the element through (z, r) plane in $d\theta$

$$Q_\phi = -k\,(dr\,dz)\,\frac{\partial T}{r\,\partial \phi}\,d\theta$$

Heat leaving

$$Q_{\phi+d\phi} = Q_\phi + \frac{\partial}{r\,\partial\phi}(Q_\phi)\,r\,d\phi$$

Net heat conducted

$$Q_\phi - Q_{\phi+d\phi} = -\frac{\partial}{r\,\partial\phi}(Q_\phi)\,r\,d\phi$$

$$= -\frac{\partial}{r\partial\phi}\left(-k\,(dr\,dz)\,\frac{\partial T}{r\partial\phi}\,d\theta\right) r\,d\phi$$

$$= k\,\frac{\partial}{\partial\phi}\left(\frac{1}{r}\,\frac{\partial T}{\partial\phi}\right)(dr\,d\phi\,dz)\,d\theta$$

$$= k\left(\frac{1}{r^2}\,\frac{\partial^2 T}{\partial\phi^2}\right)(dr\,r\,d\phi\,dz)\,d\theta \quad \text{—— (4)}$$

Add (2)+(3)+(4)

$\Rightarrow$ Net heat conducted into element from all co-ordinate directions

$$= k\,\frac{\partial^2 T}{\partial z^2}(dr\,r\,d\phi\,dz)\,d\theta + k\,(dr\,r\,d\phi\,dz)\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}\right)d\theta + k\left(\frac{1}{r^2}\,\frac{\partial^2 T}{\partial\phi^2}\right)(dr\,r\,d\phi\,dz)\,d\theta$$

$$= k\,(dr\,r\,d\phi\,dz)\,d\theta\left(\frac{\partial^2 T}{\partial z^2}+\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\,\frac{\partial T}{\partial r}+\frac{1}{r^2}\,\frac{\partial^2 T}{\partial\phi^2}\right)$$

$$= k\,(dr\,r\,d\phi\,dz)\,d\theta\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\,\frac{\partial T}{\partial r}+\frac{1}{r^2}\,\frac{\partial^2 T}{\partial\phi^2}+\frac{\partial^2 T}{\partial z^2}\right) \quad \text{—— (5)}$$

Heat generated within the element:-

$$Q = \dot{q}\,(dr\,r\,d\phi\,dz)\,d\theta \quad \text{—— (6)}$$

Heat stored in the element:-

$$= \rho\,(dr\,r\,d\phi\,dz)\,C_p\,\frac{\partial T}{\partial\theta}\times d\theta \quad \text{—— (7)}$$

Substituting eqn (5), (6), (7) in equation (1)

$$\Rightarrow k\,(dr\,r\,d\phi\,dz)\,d\theta\left[\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\,\frac{\partial T}{\partial r}+\frac{1}{r^2}\,\frac{\partial^2 T}{\partial\phi^2}+\frac{\partial^2 T}{\partial z^2}\right]+\dot{q}\,(dr\,r\,d\phi\,dz)\,d\theta = \rho\,(dr\,r\,d\phi\,dz)\,C_p\,\frac{\partial T}{\partial\theta}\times d\theta.$$

$\div$ by $(dr\,r\,d\phi\,dz)\,d\theta$

$$\Rightarrow k\left[\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\,\frac{\partial T}{\partial r}+\frac{1}{r^2}\,\frac{\partial^2 T}{\partial\phi^2}+\frac{\partial^2 T}{\partial z^2}\right]+\dot{q} = \rho\,C_p\,\frac{\partial T}{\partial\theta}$$

$$\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}\right] + \frac{\dot{q}}{k} = \frac{\rho C_p}{k}\frac{\partial T}{\partial \theta} \quad — (8)$$

Eqn (8) known as three dimensional heat conduction equation in cylindrical co-ordinates.

$$\Rightarrow \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \theta}$$

$$\left(\because \alpha = \frac{K}{\rho C_p}\right)$$

<u>Case(i)</u>

Steady flow one dimensional eqn,

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = 0$$

(or)

$$\frac{1}{r}\frac{d}{dr}\left(r \cdot \frac{dT}{dr}\right) = 0.$$

<u>Case(ii)</u>

Steady flow two dimensional eqn.

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} = 0$$

<u>Case(iii)</u>

Steady flow 3D eqn.

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} = 0.$$