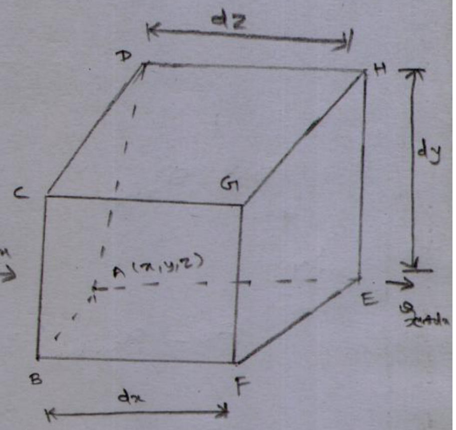
Governing equations in cartesian spherical co-ordinates:-

consider a small rectangular element of sides dx, dy and dz as shown in fig.



$$\begin{Bmatrix}\text{Net heat conducted} \\ \text{into element from} \\ \text{all co-ordinate} \\ \text{directions}\end{Bmatrix} + \begin{Bmatrix}\text{heat generated} \\ \text{with in} \\ \text{element}\end{Bmatrix} = \begin{Bmatrix}\text{Heat} \\ \text{stored} \\ \text{in the} \\ \text{element}\end{Bmatrix} \quad \text{--- (1)}$$

Net heat conducted into element from all co-ordinate directions:-

Let consider $q \rightarrow$ heat flux

$Q \rightarrow$ heat conduction

The rate of heat flow in the element side of ABCD:-

$$Q_x = q_x \, dy \, dz$$

$$Q_x = -k_x \frac{\partial T}{\partial x} dy \, dz \quad \text{--- (2)}$$

where,

k - thermal conductivity (W/mK)

$\frac{\partial T}{\partial x}$ - Temperature gradient (K/m)

The rate of Heat flow in the element side of EFGH:-

$$Q_{x+dx} = Q_x + \frac{\partial}{\partial x}(Q_x) \, dx$$

$$= -k_x \frac{\partial T}{\partial x} dy \, dz + \frac{\partial}{\partial x}\left(-k_x \frac{\partial T}{\partial x}\right) dx \, dy \, dz$$

$$Q_{x+dx} = -k_x \frac{\partial T}{\partial x} dy \, dz - \frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) dx \, dy \, dz \quad \text{--- (3)}$$

Substracting (2) - (3)

$$Q_x - Q_{x+dx} = -k_x \frac{\partial T}{\partial x} dy dz - \left[-k_x \frac{\partial T}{\partial x} dy dz - \frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) dx\, dy dz \right]$$

$$= -k_x \frac{\partial T}{\partial x} dy dz + k_x \frac{\partial T}{\partial x} dy dz + \frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) dx dy dz$$

$$Q_x - Q_{x+dx} = \frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) dx\, dy\, dz \quad \text{--- (4)}$$

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$$Q_y - Q_{y+dy} = \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) dx\, dy\, dz \quad \text{--- (5)}$$

$$Q_z - Q_{z+dz} = \frac{\partial}{\partial z}\left(k_z \frac{\partial T}{\partial z}\right) dx\, dy\, dz \quad \text{--- (6)}$$

Adding (4) + (5) + (6)

$$= \left[\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z \frac{\partial T}{\partial z}\right) \right] dx\, dy\, dz \quad \text{--- (7)}$$

Heat stored in the element :-

= mass × specific heat of the element × Rise in Temperature difference

$$= m\, C_p \frac{\partial T}{\partial t}$$

$$= \rho \times dx\, dy\, dz \times C_p \frac{\partial T}{\partial t}$$ (mass = density × volume)

∴ Heat stored in the element $= \rho C_p \frac{\partial T}{\partial t} dx\, dy\, dz$ --- (8)

Heat generated with in the element :-

$$Q = \dot{q}\, dx\, dy\, dz \quad \text{--- (9)}$$

substituting eqn (7), (8), (9) in eqn (1)

$$\left(\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z \frac{\partial T}{\partial z}\right)\right) dx\,dy\,dz + \dot{q}\, dx\,dy\,dz$$

$$= \rho C_p \frac{\partial T}{\partial t}\, dx\,dy\,dz$$

$\Rightarrow$ cancelling $dx\,dy\,dz$ on both sides and put

$k_x = k_y = k_z = k$

$$\Rightarrow \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) k + \dot{q} = \rho C_p \frac{\partial T}{\partial t}$$

$\div$ by k

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \dot{q}/k = \frac{\rho C_p}{k} \frac{\partial T}{\partial t}$$

$$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \dot{q}/k = \frac{1}{\alpha} \frac{\partial T}{\partial t}} \quad \text{--- (10)}$$

(where $\alpha = \frac{k}{\rho C_p}$ thermal diffusivity m^2/s.)

eqn (10) known as Governing equation of heat conduction in cartesian co-ordinates

case (i)

No heat sources:-

$$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}} \qquad \dot{q}/k = 0$$

This is known as Fourier's equation.

Case (ii) steady state conditions:-

Temperature does not change with time

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \dot{q}/k = 0 \quad \Rightarrow \nabla^2 T + \dot{q}/k = 0$$

This equations is known as Poisson's equations

In the absence of heat sources

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \quad \Rightarrow \nabla^2 T = 0.$$

This equation is known as Laplace equation.

Case (iii) one dimensional steady state heat conduction:-

$$\frac{\partial^2 T}{\partial x^2} + \dot{q}/k = 0$$

Case (iv) unsteady state one dimensional heat conduction

$$\frac{\partial^2 T}{\partial x^2} + \dot{q}/k = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

In the absense of heat source,

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

—— x —— x ——