UNIT-V

LAPLACE TRANSFORMS

INTRODUCTION :

Laplace Transformation, named after a great French Mathematician Pierre Simon De Laplace (1749– 1827) who used such transformations in the "Theory of probability".

Uses of Laplace Transformation :

1. It is used to find the solution of linear differential equations – ordinary as well as partial.
2. It helps in solving the differential equation with boundary values without finding the general solution and then finding the values of the arbitrary constants

Transformation :

A transformation is an operation which converts a mathematical expression to a different but equivalent form.

Laplace Transformation : Definition :

Let $f(t)$ be a function of t defined for $t>0$. Then the Laplace transform of $f(t)$, denoted by $L\{f(t)\}$ or $F(s)$ is defined by,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

Provided the integral exists.

Conditions for existence of Laplace transform:

(i) $f(t)$ should be continuous or piecewise continuous in the given closed interval $[a,b]$ where $a>0$.

(ii) $f(t)$ should be of exponential order.

Exponential order:

A function $f(t)$ is said to be of exponential order if,

$$\lim_{t\to\infty} e^{-st} f(t) = 0.$$

Example:

1. t^2 is of exponential order.

$$\lim_{t\to\infty} e^{-st} f(t) = \lim_{t\to\infty} e^{-st} t^2$$

$$= \lim_{t\to\infty} \frac{t^2}{e^{+st}} \quad \left[\frac{\infty}{\infty} \text{ Indeterminate form}\right.$$

$$= \lim_{t\to\infty} \frac{2t}{se^{+st}} \quad \text{apply L'Hospital rule}$$

$$= \lim_{t\to\infty} \frac{2}{s^2 e^{st}} = \frac{2}{\infty} = 0.$$

2. e^{t^2} is not of exponential order.

$$\lim_{t\to\infty} e^{-st} f(t) = \lim_{t\to\infty} e^{-st} e^{t^2}$$

$$= \lim_{t\to\infty} e^{-st+t^2}$$

$$= e^{\infty} = \infty$$

$\therefore e^{t^2}$ is not of exponential order.

Transforms of elementary functions :

① $L(1) = \frac{1}{s}$ where $s > 0$.

proof :

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$L(1) = \int_0^\infty e^{-st} . 1\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= -\frac{1}{s}(0-1) = \frac{1}{s}$$

$$\boxed{L(1) = \frac{1}{s}}$$

② $L(K) = \frac{K}{s}$.

③ $L(t) = \frac{1!}{s^2}$

$$L(t) = \int_0^\infty e^{-st} . t\, dt$$

$$= \left[\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^\infty$$

$$\boxed{L(t) = \frac{1!}{s^2}}$$

④ $L(t^2) = \frac{2!}{s^3}$

⑤ $L(t^n) = \frac{\Gamma_{n+1}}{s^{n+1}}$ if $s > 0$ & $n > -1$

$$\mathcal{L}(t^n) = \int_0^\infty e^{-st}\, t^n \, dt$$

put $x = st \Rightarrow dx = s\,dt$

$$\frac{dx}{s} = dt$$

$$\mathcal{L}(t^n) = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s}$$

$$= \int_0^\infty e^{-x} \cdot \frac{x^n}{s^{n+1}}\, dx$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n \, dx$$

$$\boxed{\mathcal{L}(t^n) = \frac{\Gamma_{n+1}}{s^{n+1}} = \frac{n!}{s^{n+1}}}$$

(6) $\mathcal{L}(e^{at}) = \frac{1}{s-a}$ if $s-a>0$.

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{-st} e^{at}\, dt$$

$$= \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$\boxed{\mathcal{L}(e^{at}) = \frac{1}{s-a} \text{ if } s-a>0}$$

(7) $\mathcal{L}(e^{-at}) = \frac{1}{s+a}$ if $s+a>0$

$$\mathcal{L}(e^{-at}) = \int_0^\infty e^{-st} e^{-at}\, dt$$

$$= \int_0^\infty e^{-(s+a)t}\, dt$$

$$\boxed{L(e^{-at}) = \frac{1}{s+a} \text{ if } s+a > 0}$$

⑧ To find $L(\cos at)$ & $L(\sin at)$:

We know $e^{i\theta} = \cos\theta + i\sin\theta$.

$$L(e^{iat}) = \frac{1}{s-ia}$$

$$= \frac{1}{s-ia} \cdot \frac{s+ia}{s+ia}$$

$$= \frac{s+ia}{s^2+a^2}$$

$$L(\cos at + i\sin at) = \frac{s}{s^2+a^2} + i\,\frac{a}{s^2+a^2}$$

Equating real & imaginary parts,

$$\boxed{L(\cos at) = \frac{s}{s^2+a^2} \qquad L(\sin at) = \frac{a}{s^2+a^2}}$$

⑨ To find $L(\sinh at)$:

$$L[\sinh at] = L\left(\frac{e^{at} - e^{-at}}{2}\right)$$

$$= \frac{1}{2} L(e^{at}) - \frac{1}{2} L(e^{-at})$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left(\frac{2a}{s^2-a^2}\right)$$

$$\boxed{L(\sinh at) = \frac{a}{s^2-a^2}} \text{ for } s^2 > a^2$$

(10) To find $L(\cosh at)$:

$$L(\cosh at) = L\left\{\frac{1}{2}\left[e^{at}+e^{-at}\right]\right\}$$

$$= \frac{1}{2}L(e^{at}) + \frac{1}{2}L(e^{-at})$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} + \frac{1}{s+a}\right\}$$

$$= \frac{1}{2}\cdot\frac{2s}{s^2-a^2}$$

$$\boxed{L(\cosh at) = \frac{s}{s^2-a^2}} \text{ for } s^2 > a^2$$

PROBLEMS :

(1) Find $L(t^8)$

$$L(t^n) = \frac{n!}{s^{n+1}}$$

$$L(t^8) = \frac{8!}{s^{8+1}} = \frac{40320}{s^9}$$

(2) Find $L(t+1)^2$

$$L[(t+1)^2] = L(t^2+2t+1)$$

$$= L(t^2) + 2L(t) + L(1)$$

$$= \frac{2!}{s^3} + \frac{2}{s^2} + \frac{1}{s}$$

(3) Find $L\left(\frac{1}{\sqrt{t}}\right)$

$$L\left(\frac{1}{\sqrt{t}}\right) = L(t^{-1/2})$$

$$= \frac{\Gamma(-1/2+1)}{s^{-1/2+1}} = \frac{\Gamma 1/2}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}}$$

④ $L(\sqrt{t})$.

$$\boxed{\Gamma_{n+1} = n\Gamma_n \ \& \ \Gamma_{1/2} = \sqrt{\pi}}$$

$$L(\sqrt{t}) = L(t^{1/2})$$

$$= \frac{\Gamma_{1/2+1}}{s^{1/2+1}} = \frac{1/2\,\Gamma_{1/2}}{s\sqrt{s}} = \frac{1/2\cdot\sqrt{\pi}}{s^{3/2}}$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}$$

⑤ $L(t^{5/2})$

$$L(t^{5/2}) = \frac{\Gamma_{5/2+1}}{s^{5/2+1}} = \frac{5/2\,\Gamma_{5/2}}{s^{7/2}}$$

$$= \frac{5/2\cdot 3/2\cdot 1/2\,\Gamma_{1/2}}{s^{7/2}} = \frac{15\sqrt{\pi}}{8s^{7/2}}$$

⑥ $L(e^{5t})$

$$L(e^{at}) = \frac{1}{s-a}$$

$$L(e^{5t}) = \frac{1}{s-5}$$

⑦ $L(e^{t})$

$$L(e^{t}) = \frac{1}{s-1}$$

⑧ $L(e^{-7t})$

$$L(e^{-7t}) = \frac{1}{s+7}$$

⑨ $L(e^{-t})$

$$L(e^{-t}) = \frac{1}{s+1}$$

(10) Find $L(\sin 5t)$

$$\boxed{L(\sin at) = \frac{a}{s^2+a^2}}$$

$$L(\sin 5t) = \frac{5}{s^2+5^2} = \frac{5}{s^2+25}$$

(11) Find $L(\cos 6t)$

$$\boxed{L(\cos at) = \frac{s}{s^2+a^2}}$$

$$L(\cos 6t) = \frac{s}{s^2+6^2} = \frac{s}{s^2+36}$$

(12) Find $L(\sin^2 2t)$

$$\sin^2 t = \frac{1-\cos 2t}{2}$$

$$L(\sin^2 2t) = L\left[\frac{1-\cos 2(2t)}{2}\right]$$

$$= \frac{1}{2} L(1-\cos 4t)$$

$$= \frac{1}{2}\left[L(1) - L(\cos 4t)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+16}\right]$$

(13) Find $L(\cos^2 3t)$

$$\cos^2 t = \frac{1+\cos 2t}{2}$$

$$L(\cos^2 3t) = L\left(\frac{1+\cos 2(3t)}{2}\right)$$

$$= \frac{1}{2} L(1+\cos 6t)$$

$$= \frac{1}{2}\left[L(1) + L(\cos 6t)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+36}\right]$$

(14) Find $L(\cos^3 2t)$

$$\cos^3\theta = \frac{1}{4}(\cos 3\theta + 3\cos\theta)$$

$$L[\cos^3 2t] = L\left[\frac{\cos 3(2t) + 3\cos(2t)}{4}\right]$$

$$= \frac{1}{4}\{L(\cos 6t) + 3L(\cos 2t)\}$$

$$= \frac{1}{4}\left\{\frac{s}{s^2+36} + 3\cdot\frac{s}{s^2+4}\right\}$$

$$= \frac{1}{4}\left\{\frac{s}{s^2+36} + \frac{3s}{s^2+4}\right\}$$

(15) Find $L(\sin^3 3t)$

$$\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$$

$$L(\sin^3 3t) = L\left[\frac{3\sin 3t - \sin 3(3t)}{4}\right]$$

$$= \frac{1}{4}\{3L(\sin 3t) - L(\sin 9t)\}$$

$$= \frac{1}{4}\left\{3\left(\frac{3}{s^2+3^2}\right) - \frac{9}{s^2+9^2}\right\}$$

$$= \frac{9}{4}\left\{\frac{1}{s^2+9} - \frac{1}{s^2+81}\right\}$$

(16) Find $L(\sin 2t \cos 3t)$.

$$\sin A\cos B = \frac{\sin(A+B) + \sin(A-B)}{2}$$

$$L(\sin 2t\cos 3t) = L\left\{\frac{\sin(2t+5t) + \sin(2t-5t)}{2}\right\}$$

$$= \frac{1}{2}\{L(\sin 5t) + L(\sin(-t)\}$$

$$= \frac{1}{2}\{L(\sin 5t) - L(\sin t)\}$$

$$= \frac{1}{2}\left\{\frac{5}{s^2+25} - \frac{1}{s^2+1}\right\}$$