## Initial Value Theorem:

If the Laplace Transforms of $f(t)$ and $f'(t)$ exist and $L[f(t)] = F(s)$, then

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s).$$

**Problem:**

Verify IVT for the function $f(t) = ae^{-bt}$.

**Soln:**

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s).$$

Here $f(t) = ae^{-bt}$

$$F(s) = L[f(t)] = L[ae^{-bt}]$$

$$= a \cdot \frac{1}{s+b}$$

LHS

$$\lim_{t\to 0} f(t) = \lim_{t\to 0} ae^{-bt} = a \quad \text{--- (1)}$$

RHS

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s \cdot \frac{a}{s+b}.$$

$$= \frac{\infty}{\infty} \text{ form}$$

Using l'hospital's rule

$$\text{RHS} = \lim_{s\to\infty} \frac{a}{1}$$

$$= a \quad \text{--- (2)}$$

From (1) & (2),

IVT is verified.

## Final Value Theorem:

If the Laplace Transforms of $f(t)$ and $f'(t)$ exist and $L[f(t)] = F(s)$, then $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

**Problem:**

Verify IVT & FVT for the function

$f(t) = 1 + e^{-t}(\sin t + \cos t)$.

$$L[f(t)] = L[1 + e^{-t}\sin t + e^{-t}\cos t]$$

$$= L[1] + L[e^{-t}\sin t] + L[e^{-t}\cos t]$$

$$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}$$

IVT

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

LHS

$$\lim_{t\to 0} f(t) = \lim_{t\to 0} \left(1 + e^{-t}(\sin t + \cos t)\right)$$

$$= 1 + e^0(0+1)$$

$$= 1+1$$

$$= 2$$

RHS

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s\left[\frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}\right]$$

$$= \lim_{s\to\infty}\left[1 + \frac{s}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}\right]$$

$$= \lim_{s\to\infty}\left[\frac{(s+1)^2+1+s+(s+1)s}{(s+1)^2+1}\right]$$

$$= \lim_{s\to\infty}\left[\frac{s^2+2s+1+\overset{s+2}{s+5}}{(s+1)^2+1}\right]$$

$$= \lim_{s\to\infty}\left[\frac{2s^2+4s+3}{(s+1)^2+1}\right] \quad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= \lim_{s\to\infty}\frac{4s+3}{2(s+1)}$$

$$= \lim_{s\to\infty}\frac{4}{2} = 2$$

$\therefore$ LHS = RHS

Hence IVT is verified.

Final Value Theorem:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

$$\text{LHS} = \lim_{t\to\infty} f(t)$$

$$= \lim_{t\to\infty}\left\{1+e^{-t}(\sin t+\cos t)\right\}$$

$$= 1+e^{-\infty} \Rightarrow 1+0 \quad (\because e^{-\infty}=0)$$

$$= 1$$

$$\text{RHS} = \lim_{s\to 0} sF(s)$$

$$= \lim_{s\to 0} s\left[\frac{1}{s}+\frac{1}{(s+1)^2+1}+\frac{s+1}{(s+1)^2+1}\right]$$

$$= \lim_{s\to 0}\left[1+\frac{s}{(s+1)^2+1}+\frac{s(s+1)}{(s+1)^2+1}\right]$$

$$= 1+\frac{0}{2}+\frac{0}{2} = 1$$

$\therefore$ LHS = RHS

Hence Final Value Theorem is Verified.