Taylor's Series

If $f(x)$ is analytic at all points inside a circle C, with its centre at the point a and radius R, then at each point x inside C,

$$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \cdots$$

put $a=0$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(a) + \cdots + \frac{x^n}{n!}f^{(n)}(a) + \cdots$$

put $x = a+h$.

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^n}{n!}f^{(n)}(a) + \cdots$$

Problems :

Expand $f(x) = \sin x$ in a Taylor's series about $x = \frac{\pi}{4}$

Solution:

Given: $f(x) = \sin x$

Taylor's series formula

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \cdots$$

here $x = \pi/4$

$f(x) = \sin x;\ f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f'(x) = \cos x;\ f'\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f''(x) = -\sin x;\ f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

$f'''(x) = -\cos x;\ f'''\left(\frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

$$\therefore f(x) = \frac{1}{\sqrt{2}} + \left(x - \frac{\pi}{4}\right)\left(\frac{1}{\sqrt{2}}\right) + \frac{\left(x-\frac{\pi}{4}\right)^2}{2!}\left(-\frac{1}{\sqrt{2}}\right) + \cdots + \frac{\left(x-\frac{\pi}{4}\right)^3}{3!}\left(-\frac{1}{\sqrt{2}}\right) + \cdots$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{\sqrt{2}}\frac{(x-\pi/4)^2}{2} - \frac{1}{\sqrt{2}}\frac{(x-\pi/4)^3}{6} + \cdots$$

$$\sin x = \frac{1}{\sqrt{2}}\left[1 + \left(x - \frac{\pi}{4}\right) - \frac{\left(x-\frac{\pi}{4}\right)^2}{2} - \frac{\left(x-\frac{\pi}{4}\right)^3}{6} + \cdots\right]$$

2) Expand $f(x)=e^x$ in a Taylor Series about $x=0$.

Soln: At the pt $x=0$

Given $f(x)=e^x \quad f(0)=e^0=1$

$f'(x)=e^x \quad f'(0)=e^0=1$

$f''(x)=e^x \quad f''(0)=1$

$f'''(x)=e^x \quad f'''(0)=1$

Taylor Series of $f(x)$ about $x=0$

$$f(x)=f(a)+\frac{x-a}{1!}f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots$$

$$f(x)=f(0)+\frac{x}{1}(1)+\frac{x^2}{2!}(1)+\frac{x^3}{3!}(1)+\cdots$$

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots$$

Binomial expansions

$(1+x)^{-1}=1-x+x^2-x^3+\cdots$

$(1+x)^{-2}=1-2x+3x^2-4x^3+\cdots$

$(1-x)^{-1}=1+x+x^2+x^3+\cdots$

$(1-x)^{-2}=1+2x+3x^2+4x^3+\cdots$

② Expand $f(x)=\frac{1}{1-x}$ in the region $|x|<1$. Using this result, Expand $\frac{1}{1+z^2}$ and $\tan^{-1}(z)$ in powers of z.

Solution:

Given: $f(x)=\frac{1}{1-x}$

$=(1-x)^{-1}$

Using binomial expansion

$(1-x)^{-1}=1+x+x^2+x^3+\cdots$

$\therefore f(x)=1+x+x^2+x^3+\cdots$

$\therefore \frac{1}{1-z}=1+z+z^2+z^3+\cdots$ — (*)

(ii) $\frac{1}{1+z^2}$

Now $\frac{1}{1+z^2}=\frac{1}{1-(-z^2)}$

put $z=-z^2$ in (*)

$\frac{1}{1+z^2}=1-z^2+z^4-z^6+\cdots$ — (*)1

(iii) $\tan^{-1}(z)$.

W.K.T $\int\frac{1}{1+x^2}dx=\tan^{-1}x$.

$\therefore$ Integrating (*)1 we have

$$\int\frac{1}{1+z^2}dz=\int 1-z^2+z^4-z^6+\cdots\, dz$$

$$\tan^{-1}(z)=z-\frac{z^3}{3}+\frac{z^5}{5}-\frac{z^7}{7}+\cdots$$

HW:

1) Expand $f(z)=\log(1+z)$ as Taylor's series about $z=0$

2) Expand $f(z)=\cos z$ as Taylor's series about $z=\frac{\pi}{3}$