APPLICATIONS OF LAPLACE TRANSFORM TO DIFFERENTIATE EQUATIONS:

If $L[f(t)] = F(s)$ then

$$L[y'(t)] = s\,L[y] - y(0)$$

$$L[y''(t)] = s^2 L[y] - s\,y(0) - y'(0)$$

① Solve the differential eqns. using LT $y''+4y'+4y=e^{-t}$ given that $y(0)=0$ & $y'(0)=0$

Soln: $y''+4y'+4y=e^{-t}$

Taking LT OBS we get,

$$L[y''+4y'+4y] = L[e^{-t}]$$

$$L[y''] + 4L[y'] + 4L[y] = \frac{1}{s+1}$$

$$[s^2 L[y] - s\,y(0) - y'(0)] + 4[s\,L[y] - y(0)] + 4L[y] = \frac{1}{s+1}$$

Gn: $y(0)=0,\ y'(0)=0$

$$\Rightarrow [s^2 L[y] - s\times 0 - 0] + 4[s\,L[y] - 0] + 4L[y] = \frac{1}{s+1}$$

$$\Rightarrow s^2 L[y] + 4s\,L[y] + 4L[y] = \frac{1}{s+1}$$

$$\Rightarrow [s^2+4s+4]\, L[y] = \frac{1}{s+1}$$

$$\Rightarrow L[y]\, [s+2]^2 = \frac{1}{s+1}$$

$$\Rightarrow L[y] = \frac{1}{(s+1)(s+2)^2}$$

$$\Rightarrow y = L^{-1}\left[\frac{1}{(s+1)(s+2)^2}\right]$$

Now $$\frac{1}{(s+1)(s+2)^2} = \frac{A}{(s+1)} + \frac{B}{(s+2)} + \frac{C}{(s+2)^2}$$

$$1 = A(s+2)^2 + B(s+2)(s+1) + C(s+1)$$

put $s=-2 \Rightarrow \boxed{C=-1}$

put $s=-1 \Rightarrow \boxed{A=1}$

put $s=0 \Rightarrow \boxed{B=-1}$

Now $$\frac{1}{(s+1)(s+2)^2} = \frac{1}{(s+1)} - \frac{1}{(s+2)} - \frac{1}{(s+2)^2}$$

$$\therefore y = L^{-1}\left[\frac{1}{(s+1)(s+2)^2}\right] = L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{s+2}\right] - L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$= e^{-t} - e^{-2t} - te^{-2t}$$

2) Solve using LT $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 2e^{-3t}$, $y(0)=1$, $y'(0)=-2$.

Soln: $y'' + 6y' + 9y = 2e^{-3t}$

Taking LT OBS we get,

$$L[y'' + 6y' + 9y] = 2L[e^{-3t}]$$

$$L[y''] + 6L[y'] + 9L[y] = 2L[e^{-3t}]$$

$$\{s^2L[y] - sy(0) - y'(0)\} + 6\{sL[y] - y(0)\} + 9L[y] = \frac{2}{s+3}$$

$$\{s^2L[y] - s + 2\} + 6\{sL[y] - 1\} + 9L[y] = \frac{2}{s+3}$$

$$\Rightarrow s^2L[y] + 6sL[y] + 9L[y] - (s+4) = \frac{2}{s+3}$$

$$(s^2+6s+9)L[y] = \frac{2}{(s+3)} + (s+4)$$

$$(s+3)^2L[y] = \frac{2}{(s+3)} + (s+4)$$

$$L[y] = \frac{2}{(s+3)^3} + \frac{s+4}{(s+3)^2}$$

$$\Rightarrow y = L^{-1}\left[\frac{2}{(s+3)^3} + \frac{s+4}{(s+3)^2}\right]$$

$$= 2L^{-1}\left[\frac{1}{(s+3)^3}\right] + L^{-1}\left[\frac{(s+3)+1}{(s+3)^2}\right]$$

$$= 2L^{-1}\left[\frac{1}{(s+3)^3}\right] + L^{-1}\left[\frac{1}{s+3} + \frac{1}{(s+3)^2}\right]$$

$$= 2L^{-1}\left[\frac{1}{(s+3)^3}\right] + L^{-1}\left[\frac{1}{s+3}\right] + L^{-1}\left[\frac{1}{(s+3)^2}\right]$$

$$= 2e^{-3t} L^{-1}\left[\frac{1}{s^3}\right] + L^{-1}\left[\frac{1}{s+3}\right] + L^{-1}\left[\frac{1}{(s+3)^2}\right]$$

$$= 2e^{-3t} t^2 + e^{-3t} + te^{-3t}.$$

Solve the differential eqns. using LT

$y'' - 3y' + 2y = e^{2t}$, $y(0) = -3$, $y'(0) = 5$.

$A = -10$; $C = 1$, $B = 7$.

Soln:

$y(t) = -10e^{t} + 7e^{2t} + te^{2t}.$