LAPLACE TRANSFORM

Defn:

Let $f(t)$ be a function of t defined for $t>0$. Then the Laplace transform of $f(t)$, denoted by $L[f(t)]$ (or) $F(s)$, is defined by

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = F(s),$$ provided the integral exists.

Conditions for existence of LT:

(i) $f(t)$ should be cts. or piecewise cts in the given closed interval $[a,b]$ where $a>0$

(ii) $f(t)$ should be of exponential order $\left[\lim_{t\to\infty} e^{-st} f(t) = 0\right]$

Note:

(i) $\Gamma(n+1) = n! = \int_0^\infty x^n e^{-x} dx$

(ii) $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx = (n-1)!$

(iii) $e^{-\infty} = 0;$ $e^{0} = 1$

(iv) $\Gamma(n+1) = n\Gamma(n)$

Some Elementary Functions :

(1) $L[1] = \frac{1}{s}$ where $s>0$

(2) $L[k] = \frac{k}{s}$, where $s>0$ & k is a constant.

(3) $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$ if $s>0$ and n is not (+ve) integer

(4) $L[e^{at}] = \frac{1}{s-a}$ where $s-a>0$, 'a' is a constant

(5) $L[e^{-at}] = \frac{1}{s+a}$ where $s+a>0$

(6) $L[\sin at] = \frac{a}{s^2+a^2}$ where $s>0$

(7) $L[\cos at] = \frac{s}{s^2+a^2}$ where $s>0$

(8) $L[\sinh at] = \frac{a}{s^2-a^2}$ where $s^2>a^2$

(9) $L[\cosh at] = \frac{s}{s^2-a^2}$ where $s^2>a^2$.

(1) $L[1] = \frac{1}{s}$ where $s > 0$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$L[1] = \int_0^\infty e^{-st} dt$$

$$= \left.\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \frac{e^{-\infty} - e^0}{-s} = \frac{1}{s}$$

(2) $L[k] = \frac{k}{s}$

(3) $L[t] = \int_0^\infty e^{-st}\, t\, dt$

$$= \left.\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^\infty$$

$$= \frac{1!}{s^2}$$

(4) $L[t^2] = \frac{2!}{s^3}$

(6) $L[e^{at}] = \frac{1}{s-a}$ if $s-a>0$

$$L[e^{at}] = \int_0^\infty e^{-st} e^{at}\, dt$$

$$= \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$= \frac{1}{s-a} \text{ if } s-a>0$$

7) $L[e^{-at}] = \frac{1}{s+a}$ if $s+a>0$

$$L[e^{-at}] = \int_0^\infty e^{-st} e^{-at}\, dt$$

$$= \int_0^\infty e^{-(s+a)t}\, dt$$

$$= \frac{1}{s+a} \text{ if } s+a>0$$

(8) $L[\sin at]$ & $L[\cos at]$

We know $e^{i\theta} = \cos\theta + i\sin\theta$.

Now $L[e^{iat}] = \frac{1}{s-ia}$

$$= \frac{1}{s-ia} \times \frac{s+ia}{s+ia}$$

$$= \frac{s+ia}{s^2-(ia)^2}$$

$$= \frac{s+ia}{s^2+a^2}$$

$$\Rightarrow L[\cos at + i \sin at] = \frac{s+ia}{s^2+a^2}$$

$$\Rightarrow L[\cos at] + i\, L[\sin at] = \frac{s}{s^2+a^2} + i\,\frac{a}{s^2+a^2}$$

Equating real & imaginary parts, we get

$$L[\cos at] = \frac{s}{s^2+a^2}$$

$$L[\sin at] = \frac{a}{s^2+a^2}$$

Some problems in Elementary functions:

(1) Find $L[t^8]$

$$L[t^n] = \frac{n!}{s^{n+1}}$$

$$L[t^8] = \frac{8!}{s^{8+1}} = \frac{40320}{s^9}$$

(2) Find $L[(t+1)^2]$

$$L[(t+1)^2] = L[t^2+2t+1]$$
$$= L[t^2] + 2L[t] + L[1]$$
$$= \frac{2!}{s^{2+1}} + 2\cdot\frac{1}{s^{1+1}} + \frac{1}{s}$$
$$= \frac{2!}{s^3} + \frac{2}{s^2} + \frac{1}{s}$$

(3) Find the Laplace transform of $\frac{1}{\sqrt{t}}$

$$L\left[\frac{1}{\sqrt{t}}\right] = L\left[\frac{1}{t^{1/2}}\right] = L[t^{-1/2}]$$
$$= \frac{\Gamma(-1/2+1)}{s^{-1/2+1}}$$
$$= \frac{\Gamma(1/2)}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}}$$

Result: $\Gamma(1/2) = \sqrt{\pi}$

$$= \sqrt{\frac{\pi}{s}}$$

(4) Find $L[\sqrt{t}]$

$$L[\sqrt{t}] = L[t^{1/2}]$$
$$= \frac{\Gamma(1/2+1)}{s^{1/2+1}}$$
$$= \frac{1/2\,\Gamma(1/2)}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}$$