CAUCHY'S RESIDUE THEOREM:

If $f(z)$ is analytic at all points inside and on a simple closed curve C except at a finite number of points $z_1, z_2, z_3, \ldots z_n$ inside c then

$$\int_C f(z)\,dz = 2\pi i \left[\text{sum of residues of } f(z) \text{ at } z_1, z_2 \ldots z_n\right]$$

① Evaluate $\displaystyle\int_C \frac{e^2}{(z+2)(z+1)^2}\,dz$ where c is the circle $|z|=3$.

__Soln:__ Let $f(z) = \left[\dfrac{e^2}{(z+2)(z+1)^2}\right]$

The poles of $f(z)$ are given by

$$(z+2)(z+1)^2 = 0$$

$\Rightarrow z=-1$ is a pole of order 2 &

$z=-2$ is a pole of order 1

Gn: $|z| = 3$.

$z = -1 \Rightarrow |z| = |-1| = 1 < 3$, lies inside C.

$z = -2 \Rightarrow |z| = |-2| = 2 < 3$, lies inside C.

$$\left\{ \text{Res } f(z) \right\}_{z=-2} = \underset{z \to -2}{lt} \, (\cancel{z+2}) \cdot \frac{e^2}{(\cancel{z+2})(z+1)^2}$$

$$= \frac{e^2}{(-2+1)^2}$$

$$= e^2$$

$$\left\{ \text{Res } f(z) \right\}_{z=-1} = \underset{z \to -1}{lt} \, \frac{1}{1!} \frac{d}{dz} \left[(\cancel{z+1})^2 \frac{e^2}{(\cancel{z+1})^2 (z+2)} \right]$$

$$= \underset{z \to -1}{lt} \, \frac{d}{dz} \frac{e^2}{(z+2)}$$

$$= \underset{z \to -1}{lt} \, \frac{-e^2}{(z+2)^2}$$

$$= \frac{-e^2}{(-1+2)^2}$$

$$= -e^2.$$

$\therefore$ By cauchy's Residue theorem,

$$\int_C f(z)\,dz = 2\pi i\,[\text{Sum of the residues of } f(z) \text{ at the poles which lie inside } C]$$

$$\therefore \int_C \frac{e^2}{(z+2)^2(z+1)^2}\,dz = 2\pi i\,[e^2 - e^2] = 0$$

③ Evaluate $\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz$ where C is the circle $|z| = 3/2$

Soln: Let $f(z) = \frac{4-3z}{z(z-1)(z-2)}$

The poles of $f(z)$ are given by

$z(z-1)(z-2) = 0 \Rightarrow z=0,\ z=1,\ z=2$ are poles of order 1

$z=0 \Rightarrow |z| = 0 < 3/2$, lies inside C.

$z=1 \Rightarrow |z| = 1 < 3/2$, lies inside C.

$z=2 \Rightarrow |z| = 2 > 3/2$, lies outside C.

$$\{\text{Res } f(z)\}_{z=0} = \lim_{z\to 0} (z-0)\frac{4-3z}{z(z-1)(z-2)}$$

$$= \frac{4}{(-1)(-2)} = 2.$$

$$\{\text{Res } f(z)\}_{z=1} = \lim_{z\to 1} (z-1)\frac{4-3z}{z(z-1)(z-2)}$$

$$= \frac{1}{-1} = -1.$$

$\{\text{Res } f(z)\}_{z=2} = 0$ [lies outside c]

∴ By cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i\,[\text{sum of residues of } f(z) \text{ at the poles which lie inside c}]$$

$$= 2\pi i\,[2-1]$$

$$= 2\pi i.$$