SINGULAR POINTS :-

A point $z=a$ is said to be a singular point (or) Singularity of $f(z)$ if $f(z)$ is not analytic at $z=a$.

TYPES OF SINGULAR POINTS :

(i) ISOLATED SINGULAR POINT :

A point $z=a$ is said to be an isolated singular poin of $f(z)$ if (i) $f(z)$ is not analytic at $z=a$.
(ii) $f(z)$ is analytic at all points for some neighbourhood of $z=a$.

Eg: If $f(z) = \dfrac{z}{(z-1)(z-2)}$

Then $z=1, z=2$ are isolated singular points.

(ii) POLE :

A point $z=a$ is said to be pole of $f(z)$ of order n if we can find a positive integer n such that

$$\lim_{z \to a} (z-a)^n f(z) \neq 0$$

Eg: If $f(z) = \dfrac{z-1}{(z-2)(z-3)^4}$

Then $z=2$ is a pole of order 1.
$z=3$ is a pole of order 4

iii) Essential singular point:

A singular point $z=a$ is said to be an essential singular point $f(z)$ if the Laurent's Series of $f(z)$ about $z=a$ possesses of infinite no. of terms in the principal part (terms containing negative powers).

Eg: Let $f(z) = e^{1/z}$

clearly $z=0$ is a singular point.

$$f(z) = e^{1/z} = 1 + \frac{1/z}{1!} + \frac{(1/z)^2}{2!} + \frac{(1/z)^3}{3!} + \dots\dots$$

$\therefore$ $z=0$ is an essential singular point.

(iv) REMOVABLE SINGULAR POINT:

A singular point $z=a$ is said to be a removable singular point of $f(z)$ if the Laurent's series of $f(z)$ about $z=a$ does not contain the principal part.

Eg: Let $f(z) = \frac{\sin z}{z}$

Clearly $z=0$ is a singular point.

$$f(z) = \frac{\sin z}{z} = \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right] = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} \dots$$

$\therefore z=0$ is a removable singular point.

(1) Find the nature of the singularities of $f(z) = \frac{\sin z - z}{z^3}$

Soln:

Clearly $z=0$ is a singular point.

$$f(z) = \frac{\sin z - z}{z^3} = \frac{1}{z^3}\left[\cancel{z} - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots - \cancel{z}\right]$$

$$= -\frac{1}{3!} + \frac{z^2}{5!} - \dots$$

This is the Laurent series of $f(z)$ about $z=0$ and there is no principal part.

$\therefore z=0$ is a removable singular point.

2) State the nature of the singularity of $f(z) = \sin\left(\frac{1}{z+1}\right)$

Soln: clearly $z = -1$ is a singular point.

Now $f(z) = \sin\left(\frac{1}{z+1}\right)$

$$= \left(\frac{1}{z+1}\right) - \frac{\left(\frac{1}{z+1}\right)^3}{3!} + \frac{\left(\frac{1}{z+1}\right)^5}{5!} - \cdots$$

$$= \left(\frac{1}{z+1}\right) - \frac{1}{3!}\left(\frac{1}{z+1}\right)^3 + \frac{1}{5!}\left(\frac{1}{z+1}\right)^5 - \cdots$$

This is the Laurent's series about $z = -1$ and there is infinite no. of terms in the principal part.

$\therefore z = -1$ is an essential singular point.

3) Classify the nature of the singular point of $f(z) = \frac{\tan z}{z}$

Soln: $f(z) = \frac{\tan z}{z}$, $z = 0$ is a singular point

$$= \frac{1}{z}\left[z + \frac{z^3}{3} + \cdots\right]$$

$$= 1 + \frac{z^2}{3} + \cdots$$

This is the Laurent's series of $f(z)$ about $z = 0$ & there is no principal part.

$\therefore z = 0$ is a removable singular pt.

5) Consider the function $f(z) = \frac{\sin z}{z^4}$. Find the pole & its order.

Soln:

$f(z) = \frac{\sin z}{z^4}$, $z=0$ is a singular point.

$= \frac{1}{z^4}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right]$.

$= \frac{1}{z^3} - \frac{1}{6z} + \frac{z}{120} \cdots$

$\therefore z=0$ is a pole of order 3.