TAYLOR'S SERIES :

If $f(z)$ is analytic inside a circle C with centre at $z=a$, then $f(z)$ can be expressed as,

$$f(z) = f(a) + \frac{(z-a)}{1!}f'(a) + \frac{(z-a)^2}{2!}f''(a) + \cdots + \frac{(z-a)^n}{n!}f^{(n)}(a) + \cdots$$

which is convergent at every point inside C. This is called Taylor's series of $f(z)$ about $z=a$.

NOTE:

The Taylor's series of $f(z)$ about the point $z=0$ is gn. by $f(z) = f(0) + \frac{z}{1!}f'(0) + \frac{z^2}{2!}f''(0) + \cdots + \frac{z^n}{n!}f^{(n)}(0) + \cdots$

This series is called Maclaurin's series of $f(z)$.

① Expand $f(z) = \log(1+z)$ as Taylor's series about $z=0$ if $|z|<1$.

Soln: $f(z) = \log(1+z)$ $\qquad f(0) = \log 1 = 0$

$f'(z) = \frac{1}{1+z}$ $\qquad f'(0) = \frac{1}{1} = 1$

$f''(z) = -\frac{1}{(1+z)^2}$ $\qquad f''(0) = -\frac{1}{1} = -1$

$f'''(z) = \frac{2}{(1+z)^3}$ $\qquad f'''(0) = \frac{2}{1} = 2$

Taylor's series about $z=0$ is gn. by

$$f(z) = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} f''(0) + \frac{z^3}{3!} f'''(0) + \cdots$$

$$= 0 + \frac{z}{1!}(1) + \frac{z^2}{2!}(-1) + \frac{z^3}{3!}(2) + \cdots$$

$$= \frac{z}{1!} - \frac{z^2}{2!} + \frac{2z^3}{3!} - \cdots$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$$

② Expand $f(z) = \cos z$ about $z = \pi/3$ in Taylor's series.

$f(z) = \cos z$ $\qquad f(\pi/3) = \cos \pi/3 = 1/2$

$f'(z) = -\sin z$ $\qquad f'(\pi/3) = -\sin \pi/3 = -\sqrt{3}/2$

$f''(z) = -\cos z$ $\qquad f''(\pi/3) = -\cos \pi/3 = -1/2$

$f'''(z) = \sin z$ $\qquad f'''(\pi/3) = \sin \pi/3 = \sqrt{3}/2$

Taylor's series is

$$f(z) = f(\pi/3) + \frac{(z-\pi/3)}{1!} f'(\pi/3) + \frac{(z-\pi/3)^2}{2!} f''(\pi/3) + \frac{(z-\pi/3)^3}{3!} f'''(\pi/3) + \cdots$$

$$= \frac{1}{2} + \frac{(z-\pi/3)}{1!}\left(-\frac{\sqrt{3}}{2}\right) + \frac{(z-\pi/3)^2}{2!}\left(-\frac{1}{2}\right) + \frac{(z-\pi/3)^3}{3!}\left(\frac{\sqrt{3}}{2}\right) + \cdots$$

$$= \frac{1}{2} - \frac{(z-\pi/3)}{1!}\left(\frac{\sqrt{3}}{2}\right) - \frac{(z-\pi/3)^2}{2!}\left(\frac{1}{2}\right) + \frac{(z-\pi/3)^3}{3!}\left(\frac{\sqrt{3}}{2}\right) + \cdots$$