CAUCHY'S INTEGRAL FORMULA FOR DERIVATIVES OF AN ANALYTIC FUNCTION:

If $f(z)$ is analytic inside and on a simple closed curve c and 'a' be any point inside c then,

$$\int_c \frac{f(z)}{(z-a)^2}\,dz = \frac{2\pi i}{1!} f'(a), \quad \int_c \frac{f(z)}{(z-a)^3}\,dz = \frac{2\pi i}{2!} f''(a)$$

In general, $\displaystyle\int_c \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(a)$ where the integration being taken in the anticlockwise direction around c.

Note: Any point outside c then $\displaystyle\int_c \frac{f(z)}{(z-a)}\,dz = 0$

1) Evaluate $\displaystyle\int_c \frac{\cos \pi z}{z-1}\,dz$ if c is $|z| = 2$.

Soln:

Gn. $|z| = 2$ represents a circle $x^2+y^2 = 2^2$ whose centre is origin and radius is 2.

Here $f(z) = \cos \pi z$ and $a = 1 \Rightarrow z = 1$

$\Rightarrow |z| = 1 < 2$ lies inside c

$\therefore f(a) = \cos \pi(1) = -1$

$$\therefore \int_c \frac{\cos \pi z}{z-1}\,dz = 2\pi i(-1) = -2\pi i$$

2) $\int_C \frac{dz}{(z-3)^2}$ where C is the circle $|z|=1$

Soln: Gn. $|z|=1$ represents a circle $x^2+y^2=1$ whose centre is at origin and radius is 1.

Here $f(z)=1$ and $a=3 \Rightarrow |z|=3>1$. lies outside C

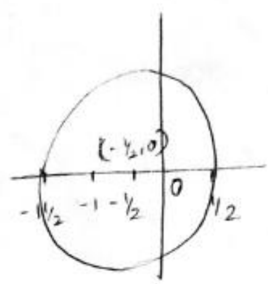
$\therefore f(a)=0$

$\therefore \int_C \frac{dz}{(z-3)^2} = 0$

) $\int_C \frac{e^z}{z+1}\,dz$ where C is the circle $|z+\frac{1}{2}|=1$

Gn: $f(z)=e^z$ and $z=-1$

$|z+\frac{1}{2}| = |-1+\frac{1}{2}| = |-\frac{1}{2}| = \frac{1}{2} < 1$, lies inside C.



$\therefore f(a) = e^{-1} = \frac{1}{e}$

$\therefore \int_C \frac{e^z}{z+1}\,dz = 2\pi i f(a) = 2\pi i \cdot \frac{1}{e}$

$= \frac{1}{e}\, 2\pi i$.

4) Use Cauchy's integral formula to evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ where C is the circle $x^2+y^2=9$.

Soln:

Gn. $x^2+y^2=9$ is the circle whose centre is origin and radius is 3 (ie) $|z|=3$.

Now $\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$

$$1 = A(z-2) + B(z-1)$$

put $z=2 \Rightarrow B=1$

put $z=1 \Rightarrow A=-1$

$$\therefore \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = -\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)} dz + \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)} dz$$

Here $f(z) = \sin \pi z^2 + \cos \pi z^2$

At $z=1$ $f(a)=f(1) = \sin \pi(1) + \cos \pi(1) = -1$

At $z=2$ $f(a)=f(2) = \sin \pi(4) + \cos \pi(4) = 1$

$$\therefore \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = -2\pi i f(1) + 2\pi i f(2)$$

$$= -2\pi i(-1) + 2\pi i(1)$$

$$= 4\pi i.$$