DEFN:

A curve is called a simple closed curve if it does not intersect itself.

CAUCHY'S INTEGRAL THEOREM (OR) CAUCHY'S FUNDAMENTAL THEOREM:

If a function $f(z)$ is analytic and its derivative $f'(z)$ is continuous at all points inside and on a simple closed curve c, then $\int_c f(z)dz = 0$

EXTENSION OF CAUCHY'S INTEGRAL THEOREM:

If $f(z)$ is analytic inside and on the annular region between the simple closed curves c and $c_1, c_2, \ldots, c_n$ then $\int_c f(z)dz = \int_{c_1} f(z)\,dz + \int_{c_2} f(z)dz + \cdots + \int_{c_n} f(z)dz$ all the integrals being taken in the anticlockwise direction.

CAUCHY'S INTEGRAL FORMULA (OR) FUNDAMENTAL FORMULA:

If $f(z)$ is analytic inside and on a simple closed curve 'c' and 'a' be any point inside c then

$\int_c \frac{f(z)}{z-a} dz = 2\pi i\, f(a)$ where the integration being taken in the +ve direction (anticlockwise direction) around c.

CAUCHY'S INTEGRAL FORMULA FOR DERIVATIVES OF AN ANALYTIC FUNCTION:

If $f(z)$ is analytic inside and on a simple closed curve c and 'a' be any point inside C then,

$$\int_c \frac{f(z)}{(z-a)^2}\,dz = \frac{2\pi i}{1!} f'(a), \quad \int_c \frac{f(z)}{(z-a)^3}\,dz = \frac{2\pi i}{2!} f''(a)$$

In general, $\int_c \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(a)$ where the integration being taken in the anticlockwise direction around c.

Note: Any point outside c then $\int_c \frac{f(z)}{(z-a)}\,dz = 0$

1) Evaluate $\int_c \frac{\cos \pi z}{z-1}\,dz$ if c is $|z| = 2$.

Soln:

Gn. $|z| = 2$ represents a circle $x^2+y^2 = 2^2$ whose centre is origin and radius is 2.

Here $f(z) = \cos \pi z$ and $a = 1 \Rightarrow z = 1$

$\Rightarrow |z| = 1 < 2$ lies inside c

$\therefore f(a) = \cos \pi(1) = -1$

$$\therefore \int_c \frac{\cos \pi z}{z-1}\,dz = 2\pi i(-1) = -2\pi i$$