Gauss Divergence theorem:

Statement:

If $\vec{F}$ is a vector point function, finite and differentiable in the region R, bounded by a closed surface S. Then the surface integral of normal component of $\vec{F}$ taken over S is equal to the integral of divergence of $\vec{F}$ to over V

$$\iint_S \vec{F}.\hat{n}\,ds = \iiint_V \nabla.\vec{F}\,dV$$, where $\hat{n}$ is the unit vector in the positive normal to S.

Problems:

① Verify gauss divergence theorem for $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ over the cube $x=0, x=1, y=0, y=1; z=0, z=1$

Sol

By gauss divergence theorem

$$\iint_S \vec{F}.\hat{n}\,ds = \iiint_V \nabla.\vec{F}\,dV$$

R.H.S

$$\nabla.\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(4xz\,\vec{i} - y^2\vec{j} + yz\vec{k})$$

$$= \frac{\partial}{\partial x} 4xz - \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(yz)$$

$$= 4z - 2y + y$$

$$= 4z - y$$

$$\iiint_V \nabla.\vec{F}\,dv = \int_0^1\int_0^1\int_0^1 (4z-y)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1 [4xz - yx]_0^1\,dy\,dz$$

$$= \int_0^1\int_0^1 (4z-y)\,dy\,dz$$

$$= \int_0^1 [4yz - y^2/2]_0^1\,dz$$

$$= \int_0^1 (4z - 1/2)\,dz = [4z^2/2 - z/2]_0^1$$

$$= 2 - 1/2$$

$$= 3/2 \quad \rightarrow ①$$

Taking L.H.S

$$\iint_S \vec{F}\cdot\hat{n}\, ds$$

Surface	$\hat{n}$	ds	Face eqn	$\vec{F}\cdot\hat{n}$
S_1 = ABCD	$\vec{i}$	dydz	x=1	4xz
S_2 = OEFG	$-\vec{i}$	dydz	x=0	-4xz
S_3 = BEFC	$\vec{j}$	dxdz	y=1	$-y^2$
S_4 = OADG	$-\vec{j}$	dxdz	y=0	y^2
S_5 = OABE	$\vec{k}$	dxdy	z=1	yz
S_6 = DEFC	$-\vec{k}$	dxdy	z=0	-yz

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$$

$$\iint_{S_1} \vec{F}\cdot\hat{n}\, ds = \int_0^1\int_0^1 4xz\, dy\, dz$$

$$= \int_0^1\int_0^1 4z\, dy\, dz \quad (\because x=1)$$

$$= \int_0^1 (4yz)_0^1 dz = \int_0^1 4z\, dz$$

$$S_1 = \left[\frac{4z^2}{2}\right]_0^1 = 4/2 = 2$$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\, ds = \int_0^1\int_0^1 -4xz\, dy\, dz$$

$$S_2 = 0 \quad (\because x=0)$$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\, ds = \int_0^1\int_0^1 -y^2\, dx\, dz$$

$$= -\int_0^1\int_0^1 dx\, dz \quad (\because y=1)$$

$$= -\int_0^1 [x]_0^1 dz$$

$$S_3 = -\int_0^1 dz = -1$$

$$\iint_{S_4} \vec{F}\cdot\hat{n}\, ds = \int_0^1\int_0^1 y^2\, dx\, dz$$

$$= 0 \quad (\because y=0)$$

$$\iint_{S_5} \vec{F}\cdot\hat{n}\, ds = \int_0^1\int_0^1 yz\, dx\, dy \quad (\because z=1)$$

$$= \int_0^1\int_0^1 y\, dx\, dy$$

$$\iint_{S_5} \vec{F}\cdot\hat{n}\, ds = \int_0^1 y\, dy$$

$$= 1/2$$

$$\iint_{S_6} \vec{F}\cdot\hat{n}\, ds = \int_0^1\int_0^1 -yz\, dx\, dy = 0$$

$$\iint_S F\cdot\hat{n}\, ds = 2 + 0 - 1 + 0 + 1/2 + 0$$

$$= 3/2 \rightarrow (2)$$

From (1) & (2)

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dv$$

Hence proved