(ii) $W = cz$.

(Magnification/Rotation)

1) Determine the region D' of w-plane into which the triangular region D enclosed by the lines $x=0$; $y=0$; $x+y=1$ is transformed under the transformation

$W = 2z$.

Soln:

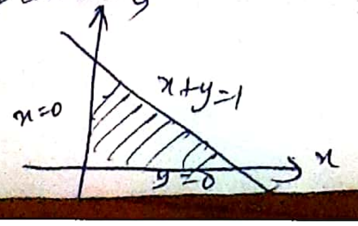
Gn: $W = 2z$.

$$u+iv = 2(x+iy)$$

$$x+iy = \frac{u+iv}{2}$$

$$x = \frac{u}{2} \text{ and } y = \frac{v}{2}$$

z-plane



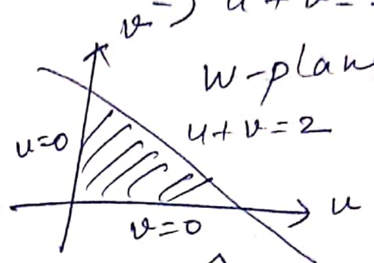
when $x=0 \Rightarrow u=0$

$y=0 \Rightarrow v=0$

$$x+y=1 \Rightarrow \frac{u}{2}+\frac{v}{2}=1$$

$$\Rightarrow u+v=2$$

w-plane.



$\therefore$ The gn line in z-plane is transformed into another triangle with $u=0$; $v=0$ and $u+v=2$ as boundaries in the w-plane.

(iii) $W = \frac{1}{z}$

1) Find the image of $|z-2i|=2$ under the transformation $w = 1/z$.

Soln:

Gn: $W = \frac{1}{z}$

$$z = \frac{1}{w}$$

$$x+iy = \frac{1}{u+iv}.$$

$$x+iy = \frac{u-iv}{u^2+v^2}$$

$$\Rightarrow x = \frac{u}{u^2+v^2}\ ;\ y = \frac{-v}{u^2+v^2}$$

Given: $|z-2i|=2$.

$$|x+iy-2i| = 2$$

$$|x+i(y-2)| = 2$$

$$\sqrt{x^2+(y-2)^2} = 2$$

$$x^2+(y-2)^2 = 4$$

$$x^2+y^2-4y+4 = 4$$

$$x^2+y^2-4y = 0$$

$$\left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2 - 4\left(\frac{-v}{u^2+v^2}\right) = 0$$

$$\frac{u^2+v^2}{(u^2+v^2)^2} + \frac{4v}{u^2+v^2} = 0$$

$$\frac{1+4v}{u^2+v^2} = 0$$

$$1+4v = 0$$

$$\Rightarrow v = -1/4$$

(which is a straight line in the w-plane.

H.W Find the image of the circle $|z-1| = 1$ in the complex plane under the mapping $w = \frac{1}{z}$.

$\boxed{u = \frac{1}{2} \text{ St-line}}$

O $\quad x$ $\quad u$

Problem (2) (iii) $w = \frac{1}{z}$ Transformation.

Find the image of the infinite strips

(i) $\frac{1}{4} < y < \frac{1}{2}$ and

(ii) $0 < y < \frac{1}{2}$

under the transformation

$w = \frac{1}{z}$.

Solution:

Given: Transformation

$$w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$$

W.K.T $w = u + iv$ $\quad x + iy = \frac{1}{u+iv}$

$z = x + iy$.

$$\therefore u + iv = \frac{1}{x+iy}$$

$$= \frac{x - iy}{(x+iy)(x-iy)}$$

$$u+iv = \frac{x-iy}{x^2+y^2}$$

$$\Rightarrow u = \frac{x}{x^2+y^2}$$

and $$v = \frac{-y}{x^2+y^2}$$

$$\frac{u}{v} = \frac{x}{-y}$$

$$x = \frac{u}{u^2+v^2}$$

$$y = \frac{-v}{u^2+v^2}$$

$$\Rightarrow x = -\frac{uy}{v} \quad \text{—— (3)}$$

$$v = \frac{-y}{\left(\frac{u^2y^2}{v^2}+y^2\right)}$$

$$= \frac{-yv^2}{u^2y^2+y^2v^2}$$

$$v = \frac{-v^2}{y(u^2+v^2)}$$

$$y = \frac{-v}{u^2+v^2} \quad \text{—— (4)}$$

(i) ~~when~~

Gn: Infinite strip

$$\frac{1}{4} < y < \frac{1}{2}$$

when $y = \frac{1}{4}$

$$\frac{-v}{u^2+v^2} = \frac{1}{4}$$

$$\Rightarrow u^2+v^2+4v = 0$$

$$\cancel{(u+v)^2} \; u^2+v^2+2(2v)+4-4=0$$

$$u^2+(v^2+2v\cdot 2+4)-4=0$$

$$u^2+(v+2)^2 = 4$$

which is an eqn of a circle, with centre $(0,-2)$ in w-plane and radius $\to 2$.

when $y = \frac{1}{2}$

$$\frac{1}{2} = \frac{-v}{u^2+v^2}$$

$$\Rightarrow u^2+(v+1)^2 = 1$$

which is an eqn of a circle with centre $(0,-1)$ in the w-plane and radius $\to 1$

Hence the infinite strip $\frac{1}{4} < y < \frac{1}{2}$ is transformed into the region common to the circles,

$u^2+(v+1)^2=1$ and $u^2+(v+2)^2=4$ in the w-plane.

$0 < y < \frac{1}{2}$

(ii). when $y=0$ and $y = \frac{1}{2}$

when $y = \frac{1}{2}$

$u^2+(v+1)^2=1$ is an eqn of the circle with centre $(0,-1)$ in the w-plane and radius 1.

when $y=0$

$$0 = \frac{-v}{u^2+v^2}$$

$-v = 0$

$v=0$, which is a straight line in the w-plane.

$\therefore$ The infinite strip $0 < y < \frac{1}{2}$ is mapped into the region outside the circle $u^2+(v+1)^2=1$ in the lower half plane.